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Diffraction by a Circular Aperture at High Frequencies

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CONTRACT No. AF 19(122)-42

SEPTEMBER, 1955

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DIFFRACTION BY A CIRCULAR APERTURE AT HIGH FREQUENCIES

Harold Levine

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Morris Kline
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Project Director

The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF 19(122)-42. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

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Abstract

An investigation is made, based on integral equations and variational principles, for the purpose of determining the first correction to the plane wave geometrical transmission cross-section of a circular aperture in an infinite plane screen. The correction is sensitive to the screen boundary condition, and in this paper details are given, for normal incidence only, for the acoustically 'soft' screen, where the wave function vanishes. There are two ways of proceeding; 1) to use equations which characterize the normal derivative of the wave function at the screen, and 2) to start with the wave function itself in the aperture. As a consequence of approximations employed, there is some disagreement in the results. The screen formulation turns out to possess advantages, and predicts a cross-section

$$\sigma \simeq \pi a^2 \left(1 - \frac{2}{\sqrt{\pi}} \; \frac{\sin(2ka - \frac{\pi}{4})}{(ka)^{3/2}}\right), \qquad ka \gg 1,$$

where a is the aperture radius. In either approach, use is made of the well-known problem of diffraction by a half-plane or straight edge, a configuration which is relevant when the aperature dimension greatly exceeds the wavelength. It is shown that the straight-edge screen distribution due to a grating plane wave, and not the primary wave, plays the decisive role for the correction in the cross-section. A comparison with other approximation techniques is given, and emphasis is placed on the role of boundary conditions at the aperture rim.

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1. Introduction

The mathematical analysis of diffraction phenomena has been significantly extended in recent years. A variety of methods, new and old, have proved their worth in enlarging the roster of problems amenable to calculation, particularly as regards configurations of geometrical complexity. It should be remarked, however, that the form of solutions thus obtained may not always be well suited for describing certain features of the scattering problem, such as the high-frequency or short-wavelength characteristics. The latter can be recognized in the zero-wavelength (or geometrical optics) limit, although without the interesting distinction that is manifest at long wavelengths, say, between like scatterers of different composition. To exhibit a corresponding difference in the scattering at high frequencies one must refine the geometrical optics approximation.

By way of illustration, let us consider plane sound waves (of harmonic time dependence and infinitesimal amplitude) impinging normally on a circular aperture

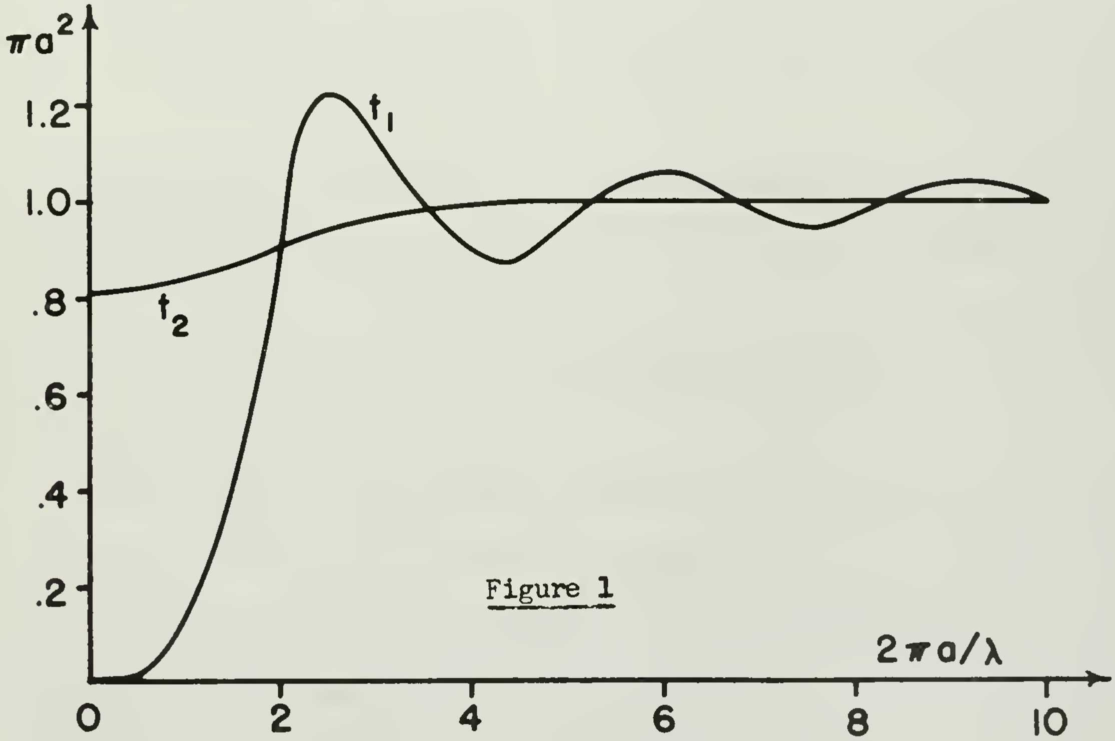

Figure 1

Transmission coeffcients for a circular aperture for normal incidence.

t_1 - acoustically 'soft' screen

t_2 - acoustically 'hard' screen

in a plane screen. In Fig. 1, the energy transmission cross-section of the aperture (relative to its area) is depicted for both an acoustically soft (I) and hard (II) screen. If the acoustical designations are interchanged, these curves also yield values of half the total scattering cross-section of a circular disk. When the wavelength λ is large compared to the aperture radius a, there is an evident contrast in the transmission cross-sections for the two screen boundary conditions. The origin of this contrast is readily traced to the different orders of magnitude for the incident field and its normal derivative, at the aperture plane; the latter is smaller by the amount $1/\lambda$. It is also clear that the boundary condition affects the manner in which the cross-sections approach the common geometrical limit ($\lambda = 0$). The exact series representation of each cross-section[1,2], on the basis of which of the curves in Fig. 1 are sketched, lends itself to approximation at long wavelengths. However, such representations prove unsuitable at short wavelengths, where even verification of the geometrical optics limit is analytically difficult. Any further comparison of the two boundary-value problems when $\lambda >> a$ is thus far unavailable.

To resolve features such as pointed up in the preceding discussion is one aspect of a study concerned with the behavior of solutions of the wave equation

$$(1) \qquad (\nabla^2 + k^2)\,\Psi(\vec{r}) = 0$$

for $k = \frac{2\pi}{\lambda} \to \infty$. The nature of asymptotic expansions in inverse powers of k for wave functions has been considered by Luneburg[3], Kline[4], and more recently by Keller and Friedlander[5]. These authors obtain and discuss conditions for successive terms in the asymptotic expansion, of which the first corresponds to geometrical optics. The technique is applicable in some scattering problems[6], although that of the aperture is not at present within its scope.

It is the purpose of this and a companion paper to study the high-frequency transmission properties of circular apertures in soft and hard screens, with the aim of clarifying the distinction manifest in Fig. 1, and deducing the lowest-order deviations (in reciprocal powers of ka) from the geometrical cross-section. A suitable framework for such considerations is provided by integral equations and variational principles, whose forms depend on the particular boundary-value problem; it is thus appropriate to deal separately with the two screen types. In this paper we shall be concerned with the analysis of transmission through an aperture in a soft screen. The problem of the hard screen is not treated here, although some comments on it are included.*

Section 2 contains preliminary information relevant to the boundary-value problem; various high-frequency approximation procedures are described and the results previously obtained are given. Section 3 is devoted to an integral equation characterizing a screen distribution, and Section 4 to its companion variational principle. The aperture integral equation is then considered (Section 5), and finally an alternative technique for the screen basis is presented (Appendix 1).

It may be expected that similar procedures are appropriate for other configurations, although the special simplicity of plane scatterers will become apparent. It should be noted that the present method has the advantage of dispensing with specific functions related to the geometry under consideration (spheroidal functions for the circular aperture[1,2]).

---

* The analogous problem involving plane electromagnetic waves and a perfectly conducting screen has been treated by H. Chang in a thesis at Harvard University (1955).

2. Description of methods and results

Consider the infinitesimally thin screen S, located in the plane $z = 0$, with an aperture A of finite extension. The scalar plane wave e^{ikz} is assumed to be incident on the aperture. If $\Psi(r)$ denotes the acoustic velocity potential, and obeys the wave equation (1), we have $\Psi = 0$ or $\frac{\partial \Psi}{\partial z} = 0$ on the screen according as the latter is soft or hard. In terms of the Green's function

$$(2) \qquad G(\vec{r},\vec{r}\,') = \frac{\exp\left\{ik|\vec{r} - \vec{r}\,'|\right\}}{4\pi\,|\vec{r} - \vec{r}\,'|},$$

there are two familiar representations of $\Psi(\vec{r})$ in the half-space $z > 0$:

$$(3) \qquad \Psi(\vec{r}) = 2\int_{S+A} \Psi(\vec{\rho}\,')\,\frac{\partial}{\partial z'}\,G(\vec{r},\vec{r}\,')\,\Big|_{z'=0}\, dS',$$

and

$$(4) \qquad \Psi(\vec{r}) = -2\int_{S+A} \frac{\partial}{\partial z'}\,\Psi(\vec{r}\,')\,\Big|_{z'=0}\, G(\vec{r},\vec{\rho}\,')\,dS',$$

where $\vec{r}$, $\vec{\rho}$ denote points that lie outside and on the aperture plane respectively. Thus, the transmitted wave function $\Psi(\vec{r})$ is determined when its values or those of its normal derivative are specified in the aperture and on the shadow side of the screen. Following Braunbek[7], we may tabulate *a priori* our knowledge of these boundary values for a soft screen, with the assumed excitation:

Table I

	Boundary values of Ψ		Boundary values of $\partial_z\Psi$	
	S	A	S	A
Exact	0	unknown	unknown	ik
Kirchhoff approximation	0	1	0	ik

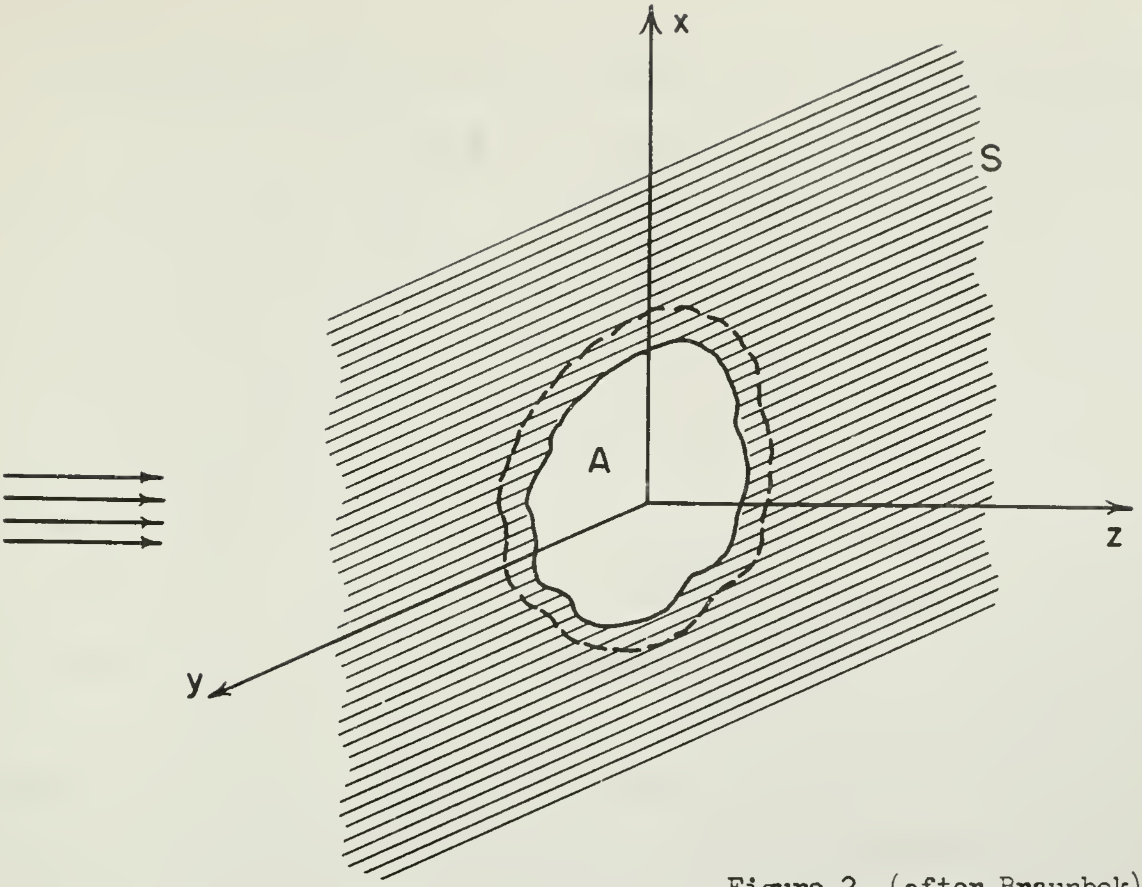

Figure 2 (after Braunbek)

That $\partial_z \Psi$ is rigorously equal to $\partial_z \Psi^{inc}$ in A is a consequence of symmetry in the geometrical regions (i.e., half-space) on opposite sides of the aperture, coupled with the infinitesimal thickness of S. It is evident that the Kirchhoff approximation assumes incident or unperturbed distributions in the aperture, and if used with the representation (4) gives rise to an unspecified error in truncating the integral beyond A. The Kirchhoff proposal for calculating the transmitted field has long been recognized as deficient in principle, except in the geometrical optics limit. When the wavelength is small compared with aperture dimensions, the normal derivative $\partial_z \Psi$ on the shadow face of the screen is significant principally in a strip region bordering the aperture, which is a few wavelengths in width (see Fig. 2). Braunbek's proposal for improving on the Kirchhoff approximation is to characterize and incorporate in (4) values of $\partial_z \Psi$ in this strip which pertain to a local infinite straight edge tangent at each point to the aperture rim. The straight-edge, or half-plane distribution for plane-wave excitation is well known from the original investigation of Sommerfeld[8], and the work of many subsequent authors.

In a similar manner, the representation (3), which involves only the aperture values of Ψ when the screen is soft, can be extended beyond its Kirchhoff approximation. To do this one must take into account an additional contribution to the integral in (3) which arises from a strip region within the aperture and near the rim, where the wave function differs significantly from its incident value; indeed, at the rim the correct wave function vanishes. The latter difference, not known exactly, is to be identified locally with the help of the aperture distribution appropriate to a half-plane whose straight edge is determined by the rim tangent.

For the aperture of circular shape, Braunbek[9] thus obtains the following expressions for the amplitude of the transmitted wave on the axis of symmetry (the z-axis), for the representations (4) and (3) respectively:

$$(5) \qquad \Psi = -ik \int_0^a \frac{e^{ikr}}{r} \rho \, d\rho - ik \int_a^\infty F(\zeta) \frac{e^{ikr}}{r} \rho \, d\rho,$$

and

$$(6) \qquad \Psi = - \int_0^a \frac{\partial}{\partial z} \left(\frac{e^{ikr}}{r}\right) \rho \, d\rho + \int_0^a \Phi(\eta) \frac{\partial}{\partial z} \left(\frac{e^{ikr}}{r}\right) \rho \, d\rho,$$

where

$$(7) \qquad \zeta = -\eta = k(\rho - a),$$

and

$$(8) \qquad F(\zeta) = \frac{1}{\sqrt{\pi}} e^{-i\pi/4} \left\{ 2 \int_{\sqrt{\zeta}}^\infty e^{i\tau^2} d\tau - \frac{i}{\sqrt{\zeta}} e^{i\zeta} \right\},$$

$$(9) \qquad \Phi(\eta) = \frac{2}{\sqrt{\pi}} e^{-i\pi/4} \int_{\sqrt{\eta}}^\infty e^{i\tau^2} d\tau .$$

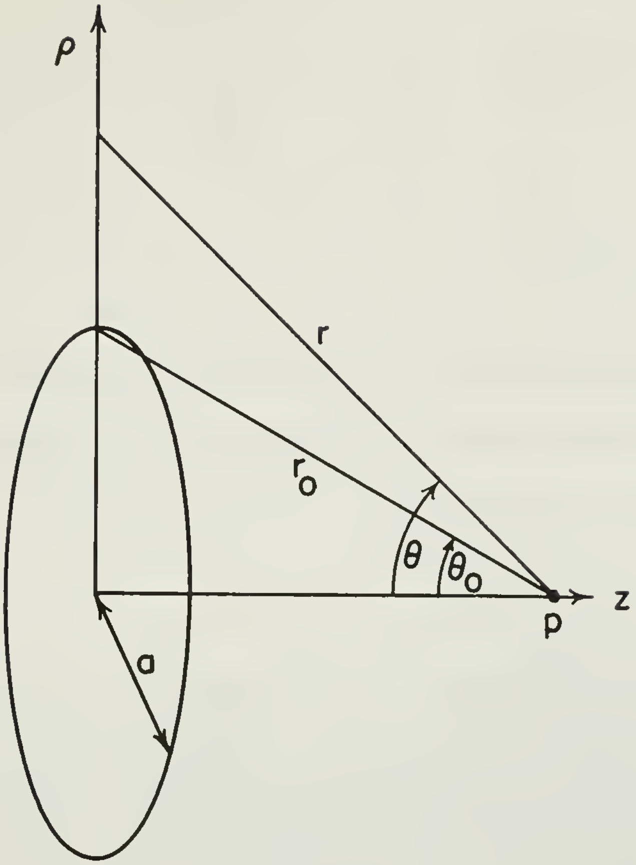

Figure 3

Here the aperture radius is a, and the geometrical significance of r and ρ can be inferred from Fig. 3. The first integrals in (5) and (6) are evidently Kirchhoff terms, although the former actually gives the exact aperture contribution (see Table I). For large values of ka, and when P is not near the aperture plane, the functions Ψ from (5) and (6) agree as regards terms of lowest order in reciprocal powers of ka; the higher terms are different and in fact outside the scope of this approximation. In lowest order, then (cf. [9]),

$$\Psi \simeq e^{ikz} - e^{ikr_o} \sqrt{1+p} \,, \qquad ka \gg 1, \tag{10}$$

where

$$p = \frac{a}{r_o} = \sin \theta_o \,.$$

It is easy to verify that if we let the point P recede from the aperture ($z \to \infty$), then the expression for Ψ goes over into a spherical wave form, namely

(11) $$\Psi \simeq A \frac{e^{ikz}}{z}, \qquad z \to \infty$$

where

(12) $$A = -\frac{ika^2}{2} - \frac{a}{2} .$$

The imaginary part of the amplitude (12) can be traced to the Kirchhoff integrals in (5) and (6), while the real part has its origin in the straight-edge corrections. Now, if we use the well-known theorem[10]:

(13) $$\sigma = -\frac{2\pi}{k} \text{ Im } A,$$

which connects the plane-wave transmission cross-section of the aperture (for normal incidence here) with the imaginary part of the forward scattered spherical wave amplitude, it is evident that in the approximation (12) we have

(14) $$\sigma = \pi a^2 ,$$

the geometrical result. Thus, while the Braunbek procedure improves the scattering amplitude relative to the Kirchhoff theory, the relation (13) shows that the correction is not sufficient to alter the cross-section.* The cross-section can naturally be identified with the integrated energy flux over all directions in the half-space $z > 0$; however, as Braunbek indicates, there is some difficulty in obtaining from (5) or (6) a form of the scattered intensity in the radiation field which holds uniformly for all directions, and consequently he gives no explicit correction to (14).

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* From the higher terms in $1/ka$ that the representations (5) and (6) yield, one can obtain frequency-dependent additions to (14) via (13), but these lack quantitative significance; thus, the succeeding term is $O((ka)^{-2})$, whereas we shall prove that it should actually be $O(ka)^{-3/2})$.

Since the cross-section theorem has obvious merit for computational purposes, we use it with more accurate evaluations of the forward amplitude A, which may be accomplished by improving on the Braunbek distributions in the plane of the screen. The lowest-order correction to the cross-section (14) is in fact obtainable after extending the Braunbek screen distribution.* This extension is also derived from a straight-edge configuration, where now the excitation arises from a secondary plane wave at grazing incidence.** The grazing wave describes an interaction between points at roughly opposite ends of an aperture diameter; that this is consistent with a correction to the geometrical cross-section can be inferred from a ray description, according to which undeviated rays passing through the aperture provide the geometrical part, whereas diffraction effects originate with rays scattered from the rim across the aperture and thence along the incident direction. Higher-order corrections evidently refer to multiple scattering of rays across the aperture.

The corresponding description of the aperture distribution is rather more difficult to obtain, although it is clear that the straight-edge variation near the rim, as generated by the incident wave, does not provide the interference or standing-wave character inherent to the proper aperture distribution. An accurate form of the aperture distribution at short wavelengths, both near the rim and at the center, would seem to require more powerful integral equation techniques.

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* More precisely, there is also a geometrical factor that modulates the straight-edge function, with maximum value unity at the edge.

** S. Karp and A. Russek have considered this kind of interaction in the case of an infinite slit formed by two half-planes; see Research Report EM-75, New York University, Division of Electromagnetic Research.

Another approach to the determination of the scattered amplitude is through variational principles[10]. Here, however, we are confronted by the formidable nature of the variational integrals involved when the Braunbek or more refined distributions are utilized. However, the integrals can be evaluated approximately when the Green's function occurring therein is modified in a manner suggested by the derivation of such distributions. For instance, the lowest-order correction to (14) emerges from a variational principle with use of only the Braunbek screen distribution, if the Green's function is suitably characterized. It is noteworthy that even a first correction to the geometrical cross-section obtained by any method proves reliable at moderate frequencies, a feature well known in other instances of asymptotic developments. This circumstance contrasts markedly with the behavior at low frequencies, where successive terms in a power series expansion do not significantly extend the qualitative or quantitative aspects of the cross-section.

The explicit correction to the geometrical optics cross-section (14) is of interest in itself, and also as a basis for comparison between different methods of approach. Our calculations, given in Sections 3 and 4, which are based on the representation (4), yield *

$$(15) \qquad \sigma \simeq \pi a^2 \left(1 - \frac{2}{\sqrt{\pi}} \; \frac{\sin(2ka - \frac{\pi}{4})}{(ka)^{3/2}}\right), \qquad ka \gg 1$$

as the leading terms of an asymptotic development in reciprocal powers of ka. In deriving the frequency-dependent term of (15), which is believed to be accurate, the normal derivative of the wave function has the proper geometrical behavior at

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* This result was contained in an abstract by the author for the Symposium on Microwave Optics, McGill University, Montreal, Canada, June 1953.

the screen near the rim (where it tends to infinity as the inverse square root of the distance), and in the aperture. The importance of suitable behavior for the screen distribution, which covers an infinite domain, is pointed up by inadequacy of the following (Kirchhoff type) approximation: The normal derivative at the screen is computed from the representation (3), when aperture and incident values of the wave function are identified. The resulting screen distribution, with a stronger rim singularity, will be shown in Section 4 to imply a deviation of $O((ka)^{-1})$ from the geometrical optics cross-section. But the geometrical behavior near the rim is not sufficient in itself to ensure the proper deviation from the geometrical optics cross-section, as evidenced by previously described application of the Braunbek distribution and cross-section theorem.

Aperture distributions at high frequencies can be approximated in several ways, although these do not reproduce the expression (15). It is significant that the form of (15) can be achieved with an aperture wave function that does not vanish at the rim, as is characteristic of the proper function which varies locally as the square root of the intervening distance. This feature can be verified in the framework of another Kirchhoff type calculation which involves both the representation (3) and (4). Thus, from (3), it follows rigorously that

$$A = -\frac{ik}{2\pi}\int_A \Psi(\vec{\rho})\, dS, \tag{16}$$

and if now the wave function (4) is approximated in the Kirchhoff sense by confining the integral to A, viz:

$$\Psi_K(\vec{r}) = -2ik\int_A G(\vec{r},\vec{\rho}\,')\, dS', \tag{17}$$

the result of combining (16) and (17) yields

$$A_K = -\frac{k^2}{\pi}\int_A G(\vec{\rho},\vec{\rho}\,')\,dSdS'. \tag{18}$$

When the amplitude A_K (where the index K denotes Kirchhoff approximation) is employed in the cross-section theorem, it follows that

$$\sigma_K = 2k\ \mathrm{Im}\int_A G(\vec{\rho},\vec{\rho}\,')dSdS'. \tag{19}$$

For a circular aperture, the integral in (19) is readily evaluated, with the result

$$\sigma_K = \pi a^2\left(1 - \frac{J_1(2ka)}{ka}\right) \tag{20}$$

$$\simeq \pi a^2\left(1 - \frac{1}{\sqrt{\pi}}\ \frac{\sin(2ka - \frac{\pi}{4})}{(ka)^{3/2}}\right), \qquad ka >> 1, \tag{21}$$

while the aperture wave function can be expressed as *

$$\Psi_K(\rho) = ka\int_0^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\ J_0(\zeta\rho)J_1(\zeta a)\ . \tag{22}$$

Thus our result (15) for the deviation from the geometrical cross-section has a coefficient which is twice that of the foregoing Kirchhoff approximation; the associated wave function $\Psi_K(a)$ is different from zero, and in fact has the value 1/2 if the radical is replaced by k, this being the mean of the values 1

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* The same aperture distribution can be obtained if the screen is regarded as an obstacle in otherwise free space, with the complete wave function a superposition of incident and scattered parts; the discontinuity in normal derivative of the wave function at the screen, or 'source', of the scattered field need only be approximated by its value in the absence of an aperture.

and 0 for $\rho < a$, $\rho > a$, respectively. The improvement in the cross-section afforded by the factor 2, even at moderately small values of ka, is evident if we compare the numerical values from (21) with the exact calculations[1,2] (see Table II). A reduction of the gap between (15) and (21) is achieved if we augment (17) by a screen integral with the primary straight-edge distribution; it turns out that the coefficient of the second term in (21) is then increased by a factor $\sqrt{2}$.

Frahn[11] has adopted Braunbek's proposal for the aperture distribution (which vanishes at the rim), and, using (16) and the cross-section theorem, claims that

$$\sigma \simeq \pi a^2 \left(1 + \frac{1}{2\sqrt{\pi}} \frac{\sin(ka + \frac{\pi}{4})}{(ka)^{3/2}}\right), \qquad ka >> 1.$$

Unfortunately, there appears to be an error in the reduction of the cross-section to its asymptotic dependence on ka; this error invalidates the second term above, which is actually $O((ka)^{-2})$, without the trigonometric factor. Indeed, for an aperture distribution of this type, some other means of computing the cross-section is necessary to obtain a functional behavior as in (15); one possibility is furnished by a variational principle, as will be demonstrated in Section 5.

Without entering into detail here, we may appreciate the reason why high-frequency transmission characteristics of apertures differ in soft and hard screens, as typified in Fig. 1. The distinction arises from the differences in the strength of interaction 'across' the aperture: none at all is provided by a plane wave at grazing incidence (or even a remote coplanar line source) on a semi-infinite hard screen, whereas the same wave incident on a soft screen does imply such interaction. Thus, for an aperture in a hard screen, the correction to the geometrical cross-section represents a higher-order effect and appears with a larger inverse power of ka than if the screen is soft.

Table II

Transmission coefficient for circular aperture in 'soft' screen for normal incidence.

$t_1^{(L)}$ using approximate formula (15).

$t_1^{(B)}$ using exact values taken from Storruste and Bouwkamp[12].

ka	$t_1^{(L)}$	$t_1^{(B)}$
.1	624.	.000003
.2	5.74	.0000486
.3	2.27	.000250
.4	.349	.000809
.5	.320	.00203
.6	.0220	.00436
.7	-.111	.00843
.8	-.147	.0151
.9	-.122	.0255
1.0	-.0576	.0411
1.1	.0338	.064
1.2	.142	.097
1.3	.261	.142
1.4	.385	.203
1.5	.509	.283
1.6	.629	.384
1.7	.744	.505
1.8	.850	.642
1.9	.945	.783
2.0	1.029	.922
2.1	1.100	1.042
2.2	1.158	1.135
2.3	1.202	1.200
2.4	1.233	1.235
2.5	1.251	1.248
2.6	1.257	1.244
2.7	1.253	1.227
2.8	1.240	1.202
2.9	1.218	1.173
3.0	1.190	1.142
3.1	1.158	1.109
3.2	1.122	1.078
3.3	1.085	1.048
3.4	1.048	1.020
3.5	1.012	0.994
3.6	.978	0.969
3.7	.948	0.946
3.8	.923	.926
3.9	.902	.908
4.0	.887	.893
4.1	.877	.880
4.2	.873	.872
4.3	.874	.867
4.4	.879	.866
4.5	.889	.869
4.6	.907	.878
4.7	.920	.895
4.8	.939	.915
4.9	.959	.937
5.0	.979	.958
6.0	1.075	1.065
7.0	.963	.964
8.0	.976	.967
9.0	1.042	1.039
10.0	.987	.987

3. Screen integral equation

In this section we base our formulation of the boundary-value problem on the representation (4). This leads naturally to an integral equation for the normal derivative of the wave function at the screen. Designating

$$(23)\qquad ik\, K(\vec{\rho}) = -\frac{\partial}{\partial z}\Psi(x,y,0), \qquad \vec{\rho} \text{ in } S,$$

and recalling that $\partial_z\Psi(x,y,0) = \partial_z\Psi^{inc}(x,y,0)$ in A, we rewrite (4) as

$$(24)\qquad \Psi(\vec{r}) = -\frac{ik}{2\pi}\int_A \frac{e^{ik|\vec{r}-\vec{\rho}\,'|}}{|\vec{r}-\vec{\rho}\,'|}\, dS' + \frac{ik}{2\pi}\int_S K(\vec{\rho})\, \frac{e^{ik|\vec{r}-\vec{\rho}\,'|}}{|\vec{r}-\vec{\rho}\,'|}\, dS'.$$

From the requirement that the wave function vanish at the screen, the basic integral equation for $K(\vec{\rho})$ follows:

$$(25)\qquad \int_S K(\vec{\rho})\, \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\, dS' = \int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\, dS', \qquad \vec{\rho} \text{ in } S.$$

This equation holds for any aperture shape, but only for normal plane wave excitation. According to (24), the spherical wave amplitude $\mathcal{A}$ along the direction of incidence is related to $K(\vec{\rho})$ by

$$(26)\qquad \mathcal{A} = -\frac{ik}{2\pi}\left[A - \int_S K(\vec{\rho})\, dS\right],$$

and thus the aperture transmission cross-section (at normal incidence) becomes

$$(27)\qquad \sigma = A - \mathrm{Re}\int_S K(\vec{\rho})\, dS.$$

To improve on the geometrical value of σ, a distribution $K(\vec{\rho})$ on the screen is needed; information about $K(\vec{\rho})$ must evidently come from the integral equation.

For an aperture of circular shape, and excitation by a scalar plane wave at normal incidence, there is an obvious rotational symmetry about the normal (or z) axis passing through the center of the aperture, so that the screen distribution is a purely radial one. Even in this simple case the integral equation cannot be solved explicitly by techniques now available and we aim at an approximate solution for $ka \gg 1$, i.e., aperture radius large compared to the wavelength. One approach will now be described, and another is relegated to later consideration (Appendix 1).

Utilizing the integral representation

$$(28) \qquad \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|} = i\int_0^\infty \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}}\, J_0(\zeta|\vec{\rho}-\vec{\rho}\,'|), \quad \arg\sqrt{k^2-\zeta^2} = \begin{matrix} 0, & \zeta < k, \\ \frac{\pi}{2}, & \zeta > k, \end{matrix}$$

and the addition formula

$$(29) \qquad J_0(\zeta|\vec{\rho}-\vec{\rho}\,'|) = \sum_{-\infty}^{\infty} J_m(\zeta\rho)\, J_m(\zeta\rho')\, e^{im(\theta-\phi')}$$

(where ρ, ϕ are plane polar coordinates specifying $\vec{\rho}$), the angle integration in (25) may be performed, with the result

$$(30) \qquad \int_0^\infty \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}}\, J_0(\zeta\rho) \int_a^\infty K(\rho')J_0(\zeta\rho')\rho' d\rho' = a\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}}\, J_0(\zeta\rho)J_1(\zeta a), \quad \rho > a.$$

Note that when $k \to \infty$, and the radical $\sqrt{k^2-\zeta^2}$ is replaced by k, the left-hand side of (30) is $K(\rho)/k$, namely zero, since

$$(31) \qquad \int_0^\infty \zeta d\zeta\, J_0(\zeta\rho)J_0(\zeta\rho') = \frac{\delta(\rho-\rho')}{\rho} \quad .$$

Likewise the right-hand side of (30) vanishes in this case, since

$$(32) \qquad \int_0^\infty d\zeta\, J_0(\zeta\rho)J_1(\zeta a) = 0, \qquad \rho > a.$$

More precisely, for ka >> 1, the principal contributions to the ζ-integrals arise from $\zeta \approx k$, whence the Bessel function arguments are large compared with their orders. Inserting the asymptotic forms

$$(33) \qquad \begin{aligned} J_0(\zeta\rho) &\simeq \sqrt{\frac{2}{\pi\zeta\rho}}\cos(\zeta\rho - \tfrac{\pi}{4}) \\ J_1(\zeta a) &\simeq \sqrt{\frac{2}{\pi\zeta a}}\sin(\zeta a - \tfrac{\pi}{4}) , \end{aligned}$$

and

$$(34) \qquad \begin{aligned} J_0(\zeta\rho)J_0(\zeta\rho') &\simeq \frac{1}{\pi\zeta}(\rho\rho')^{-1/2}\Big(\cos\zeta(\rho-\rho') + \sin\zeta(\rho+\rho')\Big), \\ J_0(\zeta\rho)J_1(\zeta a) &\simeq -\frac{1}{\pi\zeta}(\rho a)^{-1/2}\Big(\sin\zeta(\rho-a) + \cos\zeta(\rho+a)\Big), \end{aligned}$$

we obtain from (30)

$$(35) \qquad \begin{aligned} &\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_a^\infty K(\rho')\Big[\cos\zeta(\rho-\rho') + \sin\zeta(\rho+\rho')\Big]\sqrt{\rho'}\, d\rho' \\ &= -\sqrt{a}\int_0^\infty \frac{d\zeta}{\zeta\sqrt{k^2-\zeta^2}}\Big[\sin\zeta(\rho-a) + \cos\zeta(\rho+a)\Big], \quad \rho > a . \end{aligned}$$

Introducing the variable x by a displacement transformation

$$\rho = a + x,$$

and writing K(x) for $K(\rho)$, we obtain

$$\begin{aligned}&\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^\infty K(x')\left[\cos \zeta(x-x') + \sin \zeta(x+x'+2a)\right] \sqrt{a+x'}\ \ dx' \\ &\qquad = -\sqrt{a} \int_0^\infty \frac{d\zeta}{\zeta\sqrt{k^2-\zeta^2}} \left[\sin \zeta\, x + \cos \zeta(x+2a)\right], \ x > 0.\end{aligned} \tag{36}$$

Now, the integral on the right-hand side of (36) which contains the sine factor is perfectly convergent, in fact

$$\int_0^\infty \frac{d\zeta}{\zeta\sqrt{k^2-\zeta^2}} \sin \zeta x = \frac{2}{\pi k} \int_0^{kx} H_0^{(1)}(t)dt, \qquad x > 0; \tag{37}$$

however, the integral with the cosine factor is not, owing to the $1/\zeta$ singularity at $\zeta = 0$. This defect, arising from inadequacy of the asymptotic forms (33), (34) at $\zeta = 0$, is not obviated even if we omit the cosine term (which oscillates more rapidly than does the sine term and may thus be expected to yield a smaller relative contribution), since it would then follow (via (37)) that the right-hand side of the modified integral equation has the limiting value $-\frac{2\sqrt{a}}{\pi k}$ for $k \to \infty$, whence $K(x)$ is frequency independent in the limit, and $\partial_z \Psi$ tends to infinity; this conflicts with the shadow notion.

A way out of the difficulty is provided by first rewriting the inhomogeneous term of the integral equation (3) in the numerically equivalent form (see Eq. (32))

$$a\int_0^\infty d\zeta \left[\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right] J_0(\zeta\rho)\ J_1(\zeta a),$$

and then inserting asymptotic expressions for the Bessel functions. This yields

$$\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^\infty I(x')\Big[\cos\zeta(x-x') + \sin\zeta(x+x'+2a)\Big]dx'$$

(38)

$$= -\int_0^\infty \frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right]\Big[\sin\zeta x + \cos\zeta(x+2a)\Big],\ x > 0,$$

where

(39)
$$I(x) = K(x)\sqrt{1+\frac{x}{a}}\ .$$

It is useful to note that

(40)
$$\int_0^\infty \frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right]\sin\zeta x = \frac{1}{2i}\int_{-\infty}^\infty \frac{d\zeta}{\zeta+i\varepsilon}\ \frac{e^{i\zeta x}}{\sqrt{k^2-\zeta^2}}\ ,\ \varepsilon \to 0,\ x > 0;$$

hence we may recast (38) in the form

$$\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^\infty I(x')\Big[\cos\zeta(x-x') + \sin\zeta(x+x'+2a)\Big]\,dx'$$

(41)

$$= \frac{i}{2}\int_{-\infty}^\infty \frac{d\zeta}{\zeta+i\varepsilon}\ \frac{e^{i\zeta x}}{\sqrt{k^2-\zeta^2}} - \int_0^\infty \frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right]\cos\zeta(x+2a),\ x > 0.$$

The inhomogeneous terms are now well defined and, as will become clear, they have suitable behavior for $k \to \infty$.

Equation (41) is fundamental to our calculation of transmission through a circular aperture in a soft screen. While the equation does not admit a solution in closed form, it is suitable for the purpose of establishing the first frequency dependent correction to the geometrical cross-section, since without the terms containing the rapidly varying trigonometric factors and the aperture radius a, an explicit solution is possible. We recognize the solution as pertaining to the shadow distribution on a half-plane, located on the positive x-axis. With this information a perturbation procedure is readily applied to (41) for the next approximation; this leads to another half-plane problem with grazing incidence, as discussed previously.

We consider first the straight edge problem with normally incident excitation, using the technique of complex Fourier integrals to solve the relevant integral equation

$$\begin{aligned}\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^\infty I^{(0)}(x')\cos\zeta(x-x')dx' &= \frac{1}{2}\int_{-\infty}^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^\infty I^{(0)}(x')e^{i\zeta(x-x')}dx' \\ &= \frac{\pi}{2}\int_0^\infty I^{(0)}(x')H_0^{(1)}(k|x-x'|)dx' = \frac{i}{2}\int_{-\infty}^\infty \frac{d\zeta}{\zeta+i\varepsilon}\,\frac{e^{i\zeta x}}{\sqrt{k^2-\zeta^2}}, \quad x > 0.\end{aligned}\tag{42}$$

Note that when $x > 0$,

$$\begin{aligned}\int_{-\infty}^\infty \frac{d\zeta}{\zeta+i\varepsilon}\,\frac{e^{i\zeta x}}{\sqrt{k^2-\zeta^2}} &= \frac{2}{ik}\int_1^\infty \frac{dt}{t}\,\frac{e^{ikxt}}{\sqrt{t^2-1}} = \frac{\pi}{ik}\int_{kx}^\infty H_0^{(1)}(t)dt \\ &\simeq \frac{1}{k}\sqrt{\frac{2\pi}{kx}}\;\exp\left[i\left(kx-\frac{\pi}{4}\right)\right], \qquad kx \to \infty, \quad -\varepsilon \to 0,\end{aligned}\tag{43}$$

whereas for $x < 0$,

$$\int_{-\infty}^\infty \frac{d\zeta}{\zeta+i\varepsilon}\,\frac{e^{i\zeta x}}{\sqrt{k^2-\zeta^2}} = -\frac{2\pi i}{k}e^{\varepsilon x} + \frac{2i}{k}\int_1^\infty \frac{dt}{t}\,\frac{e^{-ikxt}}{\sqrt{t^2-1}} = -\frac{2\pi i}{k}e^{\varepsilon x} - \frac{\pi}{ik}\int_{k|x|}^\infty H_0^{(1)}(t)dt;\tag{44}$$

retaining $\varepsilon \neq 0$ in the exponential in (44) ensures that the entire function vanishes for $x \to -\infty$.

Let us extend (42) by defining an equation for $-\infty < x < \infty$, namely

$$(45) \qquad \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_{-\infty}^{\infty} I^{(0)}(x')e^{i\zeta(x-x')}dx' = i\int_{-\infty}^{\infty} \frac{d\zeta}{\zeta+i\varepsilon} \frac{e^{i\zeta x}}{\sqrt{k^2-\zeta^2}} + F(x),$$

with the stipulations

$$(46) \qquad I^{(0)}(x) = 0, \quad x < 0, \qquad F(x) = 0, \quad x > 0.$$

It is clear that for small negative values of x, the function F(x) is proportional to the wave function off the half-plane, i.e., the aperture field; in fact,

$$F(x) \sim \sqrt{1 + \frac{x}{a}}\ \Psi(x).$$

Multiplying each term in (45) by $e^{-i\zeta' x}$ and integrating over all x, we arrive at the Fourier transform relation (again expressed in terms of a variable ζ)

$$(47) \qquad \frac{I^{(0)}(\zeta)}{\sqrt{k^2 - \zeta^2}} = \frac{i}{\zeta+i\varepsilon} \frac{1}{\sqrt{k^2-\zeta^2}} + \frac{1}{2\pi} F(\zeta),$$

where

$$(48) \qquad I^{(0)}(\zeta) = \int_{0}^{\infty} e^{-i\zeta x}\, I^{(0)}(x)dx,$$

and

$$(49) \qquad F(\zeta) = \int_{-\infty}^{0} e^{-i\zeta x}\, F(x)dx.$$

From the obvious physical inference that $I^{(0)}(x)$ has a propagation or wave vector of magnitude k in the positive x-direction, it follows via (48) that the transform $I^{(0)}(\zeta)$ is regular if $\mathrm{Im}\,\zeta < \mathrm{Im}\,k$ ($= \varepsilon$, say). The region of regularity of $F(\zeta)$, determined by (44) and (45), in conjunction with the asymptotic behavior of the Hankel function $H_0^{(1)}(k|x-x'|)$ as $x \to -\infty$, is $\text{Im}\,\zeta > -\text{Im}\,k$. We may thus regard (47) as a meaningful relation in the strip $|\text{Im}\,\zeta| < \text{Im}\,k$ of the ζ-plane. The transform equation (47) is next rewritten (in accordance with the Wiener-Hopf technique) so as to isolate terms regular in the upper ($\text{Im}\,\zeta > -\text{Im}\,k$) and lower ($\text{Im}\,\zeta < \text{Im}\,k$) overlapping half-planes, viz:

$$(50) \qquad \frac{I^{(0)}(\zeta)}{\sqrt{k-\zeta}} - \frac{i}{\zeta+i\varepsilon}\left[\frac{1}{\sqrt{k-\zeta}} - \frac{1}{\sqrt{k+i\varepsilon}}\right] = \frac{i}{\zeta+i\varepsilon}\,\frac{1}{\sqrt{k+i\varepsilon}} + \frac{1}{2\pi}\sqrt{k+\zeta}\;F(\zeta).$$

(lower half-plane) (upper half-plane)

An entire function is thereby defined throughout the ζ-plane, and this function proves to be zero, by examination of its asymptotic behavior in the respective half-planes, using the relations $I^{(0)}(\zeta) \sim (i\zeta)^{-1/2}$, $|\zeta| \to \infty$, $\text{Im}\,\zeta < 0$, and $F(\zeta) \sim (-i\zeta)^{-1}$, $|\zeta| \to \infty$, $\text{Im}\,\zeta > 0$. Thus, proceeding to the limit, $\text{Im}\,k = \varepsilon = 0$, we deduce that

$$(51) \qquad I^{(0)}(\zeta) = \frac{i}{\zeta}\left(1 - \sqrt{1 - \frac{\zeta}{k}}\right)$$

$$(52) \qquad = \frac{i}{2k} + \frac{i}{8}\,\frac{\zeta}{k^2} + \ldots$$

is the transform solution of the integral equation (42). By inversion,

$$(53) \qquad I^{(0)}(x) = \sqrt{1 + \frac{x}{a}}\,K^{(0)}(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{d\zeta}{\zeta}\,e^{i\zeta x}\left(1 - \sqrt{1 - \frac{\zeta}{k}}\right),$$

where the path of integration passes under the branch point at $\zeta = k$. On deforming the contour so as to encircle a branch cut running horizontally to the right from $\zeta = k$, it follows readily that

$$I^{(0)}(x) = \frac{1}{\pi\sqrt{k}} \int_k^\infty \frac{e^{i\zeta x}}{\sqrt{\zeta - k}} \left(1 - \frac{k}{\zeta}\right) d\zeta = \frac{e^{i\pi/4}}{\sqrt{\pi}} \left\{ \frac{e^{ikx}}{\sqrt{kx}} + i \int_{kx}^\infty \frac{e^{it}}{\sqrt{t}}\, dt \right\}$$

(54)

$$= \frac{e^{i\pi/4}}{2\sqrt{\pi}} \int_{kx}^\infty \frac{e^{it}}{t^{3/2}}\, dt,$$

from which the behavior of $I^{(0)}(x)$ for $kx \ll 1$ and for $kx \gg 1$ can be inferred. The singularity at the rim ($x = 0$) is of a familiar type.

Using this solution, we may inquire about the corresponding scattered amplitude $\mathcal{A}^{(0)}$, given in Eq. (26), which we now cast in the form

(55) $$\mathcal{A}^{(0)} = -\frac{ik}{2\pi} \left[\pi a^2 - 2\pi a \int_0^\infty I^{(0)}(x) \sqrt{1 + \frac{x}{a}}\; dx \right].$$

If the radical is approximated by $1 + \frac{x}{2a}$, we encounter the integrals (cf.(52))

(56) $$\int_0^\infty I^{(0)}(x) dx = I^{(0)}(0) = \frac{i}{2K}$$

and

(57) $$\int_0^\infty x\, I^{(0)}(x) dx = i\, I^{(0)\prime}(0) = -\frac{1}{8k^2},$$

whence

(58) $$\mathcal{A}^{(0)} \simeq -\frac{ika^2}{2} - \frac{a}{2} - \frac{i}{16k}, \qquad ka \gg 1.$$

A comparison with Eq. (12) reveals that the term $-a/2$ is indeed the Braunbek correction; the i/k term gives a contribution of order $1/k^2$ to the cross-section, but as we will show presently, a better approximation to the solution of the integral equation (41) generates a lower-order contribution, namely $k^{-3/2}$ in σ. The ease with which the scattered amplitude follows, once the transform of the screen distribution is known, will be noted.

The Fourier integral technique is not as successful in coping with the integral equation (41) for $I(x)$, since the presence of a term involving $x + x'$ in the kernel gives rise to an integral equation, rather than an algebraic relation, in the transform variable. The latter integral equation can be solved by iteration; but we shall apply instead a direct perturbation procedure to (41), which will give the desired information equally well. Let us therefore write

$$I(x) = I^{(0)}(x) + I^{(1)}(x), \tag{59}$$

where $I^{(0)}(x)$ is the function defined by (42), whose transform is contained in (51), while $I^{(1)}(x)$ is defined by the integral equation

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^{\infty} I^{(1)}(x')e^{i\zeta(x-x')}dx' &= -\int_0^{\infty} \frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right]\cos\zeta(x+2a) \\ &\quad - \int_0^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\int_0^{\infty} I^{(0)}(x')\sin\zeta(x+x'+2a)dx', \quad x > 0. \end{aligned} \tag{60}$$

The first integral on the right-hand side of (60) can be transformed according to

$$\begin{aligned} k\int_0^{\infty}\frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right]\cos\zeta(x+2a) &= \int_{k(x+2a)}^{\infty}\left\{\frac{\pi}{2}S_0(t) - i\frac{\pi}{2}J_0(t) - \frac{1}{t}\right\}dt \\ &\simeq \sqrt{\frac{\pi}{2}}\;\frac{\exp\left[ik(x+2a) - \frac{i\pi}{4}\right]}{\sqrt{k(x+2a)}}\left\{1 - \frac{5}{8}i\frac{1}{k(x+2a)}\right\} + O\Big(k(x+2a)\Big)^{-2}, \quad ka \gg 1, \end{aligned} \tag{61}$$

where S_0 denotes the Struve function of zero order. Instead of discussing the second integral in (60) separately, it is convenient to combine the two, whence, with the help of (48) and (51), we find

$$\int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^{\infty} I^{(1)}(x')e^{i\zeta(x-x')}dx'$$

$$= i\int_0^{\infty} d\zeta \left\{ e^{i\zeta(x+2a)}\left[\frac{I^{(0)}(-\zeta)}{\sqrt{k^2-\zeta^2}} + \frac{i}{\zeta}\left(\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right)\right] - e^{-i\zeta(x+2a)}\right.$$

(62)

$$\left.\left[\frac{I^{(0)}(\zeta)}{\sqrt{k^2-\zeta^2}} - \frac{i}{\zeta}\left(\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right)\right]\right\}$$

$$= \frac{1}{k}\int_0^{\infty} \frac{d\zeta}{\zeta}\left\{e^{i\zeta(x+2a)}\left(1 - \frac{1}{\sqrt{1-\frac{1}{k}}}\right) + e^{-i\zeta(x+2a)}\left(1 - \frac{1}{\sqrt{1+\frac{1}{k}}}\right)\right\}, \quad x > 0.$$

This integral equation - which differs from (42) in the structure of the inhomogeneous term - also describes a half-plane diffraction problem. Since $I^{(1)}(x)$ is viewed as a perturbation on $I^{(0)}(x)$, we may avail ourselves of a simplification, appropriate for $ka \gg 1$, in the forcing terms of (62). Now, for $\alpha \gg 1$

(63a) $$\int_0^{\infty} \frac{dt}{t}\, e^{i\alpha t}\left(1 - \frac{1}{\sqrt{1-t}}\right) \simeq -2\sqrt{\frac{\pi}{\alpha}}\,\exp\left[i\left(\alpha - \frac{\pi}{4}\right)\right] + \frac{1}{2i\alpha} + O\left(\frac{e^{i\alpha}}{\alpha^{3/2}}\right),\ ^*$$

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[*] The leading term of (63a) is easily obtained by recognizing that the integral depends principally on values of $t \approx 1$ when $\alpha \gg 1$. Hence, writing $1-t = u$, discarding the constant relative to the radical, and placing the reciprocal of t equal to unity, we find (see (93))

$$\int_0^{\infty} \frac{dt}{\sqrt{t}}\, e^{i\alpha t}\left(1 - \frac{1}{\sqrt{1-t}}\right) \simeq -e^{i\alpha}\int_{-\infty}^{1} \frac{du}{\sqrt{u}}\, e^{-i\alpha u} = -e^{i\alpha}\int_{-\infty}^{\infty} \frac{du}{\sqrt{u}}\, e^{-i\alpha u} + O\left(\frac{1}{\alpha}\right) = -2\sqrt{\frac{\pi}{\alpha}}\, e^{i\left(\alpha - \frac{\pi}{4}\right)} + O\left(\frac{1}{\alpha}\right)$$

$$(63b) \qquad \int_0^\infty \frac{dt}{t}\, e^{-i\alpha t}\left(1-\frac{1}{\sqrt{1+t}}\right) \simeq \frac{1}{2i\alpha} + O\left(\frac{1}{\alpha^2}\right),$$

so that on retaining only the lowest-order term for (62), we have

$$(64) \qquad \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^\infty I^{(1)}(x')e^{i\zeta(x-x')}dx' = -\frac{2}{k}\sqrt{\frac{\pi}{k(x+2a)}}\exp\left[ik(x+2a)-\frac{i\pi}{4}\right], \ x > 0.$$

It is apparent that (64) refers to a half-plane excited by a remote line source which is parallel to and coplanar with its edge. In the present context, this source localizes the total effect of the far side of the screen and the aperture as regards interaction in the plane of the screen. For the purpose of ascertaining the function $I^{(1)}(x)$ to the lowest order in $1/ka$, the line-source excitation in (64) can be replaced by a plane-wave excitation, namely

$$(65) \qquad \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^\infty I^{(1)}(x')e^{i\zeta(x-x')}dx' = -\frac{2}{k}\sqrt{\frac{\pi}{2ka}}\exp\left[ik(x+2a)-\frac{i\pi}{4}\right], \ x > 0.$$

The integral equations (64) and (65) are amenable to the transform technique employed earlier. For the simpler of the two cases, involving (65), we start with an extended equation

$$(66) \qquad \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_{-\infty}^\infty I^{(1)}(x')e^{i\zeta(x-x')}dx' = \left\{\begin{array}{ll} -\frac{2}{k}\sqrt{\frac{\pi}{2ka}}\exp\left[ik(x+2a)-\frac{i\pi}{4}\right], & x > 0 \\ & \\ 0 & , x < 0 \end{array}\right\} + G(x),$$

where

$$(67) \qquad I^{(1)}(x) = 0, \quad x < 0, \qquad G(x) = 0, \quad x > 0.$$

The transform equation thus becomes

$$\frac{I^{(1)}(\zeta)}{\sqrt{k^2-\zeta^2}} = \frac{1}{ik}\sqrt{\frac{1}{2\pi ka}}\;\frac{e^{2ika-i\pi/4}}{k-\zeta} + \frac{1}{2\pi}\;G(\zeta), \tag{68}$$

with

$$I^{(1)}(\zeta) = \int_0^\infty I^{(1)}(x)e^{-i\zeta x}dx, \qquad \text{regular for Im } \zeta < \text{Im } k, \tag{69}$$

$$G(\zeta) = \int_{-\infty}^0 G(x)e^{-i\zeta x}dx\;, \qquad \text{regular for Im } \zeta > -\text{ Im } k; \tag{70}$$

note that

$$\int_0^\infty e^{i(k-\zeta)x}\,dx = \frac{i}{k-\zeta}\;, \qquad \text{Im } \zeta < \text{Im } k\;. \tag{71}$$

Rearranging Eq. (68), defined in the strip $|\text{Im } \zeta| < \text{Im } k$, so as to separate terms regular in the half-planes $\text{Im } \zeta < \text{Im } k$ and $\text{Im } \zeta > -\text{ Im } k$, we have

$$\begin{aligned} &\frac{I^{(1)}(\zeta)}{\sqrt{k-\zeta}} - \frac{1}{ik}\,\frac{1}{\sqrt{2\pi ka}}\,\exp\left[i(2ka-\frac{\pi}{4})\right]\frac{\sqrt{2k}}{k-\zeta} = \\ &\qquad = \frac{1}{ik}\,\frac{1}{\sqrt{2\pi ka}}\,\exp\left[i(2ka-\frac{\pi}{4})\right]\left\{\frac{\sqrt{k+\zeta}-\sqrt{2k}}{k-\zeta}\right\} + \frac{1}{2\pi}\sqrt{k+\zeta}\;G(\zeta) = \text{Entire Function.} \end{aligned} \tag{72}$$

Since $I^{(1)}(\zeta)$ and $G(\zeta)$ have the same asymptotic behavior as $I^{(0)}(\zeta)$ and $F(\zeta)$ (cf. (50)), the entire function vanishes and consequently

$$I^{(1)}(\zeta) = \frac{1}{ik}\;\frac{1}{\sqrt{\pi ka}}\exp\left[i(2ka-\frac{\pi}{4})\right]\frac{1}{\sqrt{1-\frac{\zeta}{k}}}\;, \tag{73}$$

with

$$I^{(1)}(0) = \exp\left[i(2ka-\frac{\pi}{4})\right]\Big/\,ik\sqrt{\pi ka}\;. \tag{74}$$

The approach to Eq. (64) is similar, and instead of (66) we find

$$(75)\qquad \frac{I^{(1)}(\zeta)}{\sqrt{k^2-\zeta^2}} = -\frac{1}{k\sqrt{\pi k}}\exp\left[i(2ka-\frac{\pi}{4})\right]\int_0^{\infty}\frac{e^{i(k-\zeta)x}}{\sqrt{x+2a}}\,dx + \frac{1}{2\pi}G(\zeta)\,.$$

Integration by parts shows that

$$(76)\qquad \int_0^{\infty}\frac{e^{i(k-\zeta)x}}{\sqrt{x+2a}}\,dx = \frac{1}{\sqrt{2a}}\,\frac{1}{k-\zeta} + \frac{1}{2(2a)^{3/2}}\cdot\frac{1}{(k-\zeta)^2} + O\left((k-\zeta)^{-3}\right),\quad \mathrm{Im}\,\zeta < \mathrm{Im}\,k;$$

the first term of this equation leads back directly to (68). If the second term is also retained, the transform $I^{(1)}(\zeta)$ turns out to be

$$(77)\qquad I^{(1)}(\zeta) = \frac{1}{ik}\,\frac{1}{\sqrt{\pi ka}}\exp\left[i(2ka-\frac{\pi}{4})\right]\frac{1}{\sqrt{1-\frac{\zeta}{k}}} + $$

$$+\frac{1}{4ik}\,\frac{1}{(2\pi a)^{3/2}}\sqrt{1-\frac{\zeta}{k}}\,\exp\left[i(2ka-\frac{\pi}{4})\right]\int_{\infty}^{-\infty}\frac{dt}{t-\zeta}\,\frac{\sqrt{k+t}}{(t-k)^2}\,,$$

where the path of integration is a line in the strip $|\mathrm{Im}\,t| < \mathrm{Im}\,k$, and the integral represents (proportionally) that part of $\sqrt{k+\zeta}\,/\,(k-\zeta)^2$ which is regular in the half-plane $\mathrm{Im}\,\zeta < \mathrm{Im}\,k$. The integral may be evaluated after completion of a circuit in the upper half of the t-plane; we have

$$(78)\qquad I^{(1)}(\zeta) = \frac{1}{ik}\,\frac{1}{\sqrt{\pi ka}}\exp\left[i(2ka-\frac{\pi}{4})\right]\frac{1}{\sqrt{1-\frac{\zeta}{k}}} - \frac{1}{16k\sqrt{\pi}}\,\frac{1}{(ka)^{3/2}}\,e^{i(2ka-\frac{\pi}{4})}\,\frac{3+\frac{\zeta}{k}}{(1-\frac{\zeta}{k})^{3/2}},$$

and it is verified that the deviation from (73) is of higher order in 1/ka.

When $I^{(1)}(x)$ is derived from (73) and combined with $I^{(0)}(x)$, we find

$$(79) \qquad I(x) = I^{(0)}(x)+I^{(1)}(x) = \frac{e^{i\pi/4}}{\sqrt{\pi}} \left[\left(1-2\sqrt{\frac{\pi}{ka}}\, e^{i(2ka-\frac{\pi}{4})}\right)\frac{e^{ikx}}{\sqrt{kx}} + i\int_{kx}^{\infty} \frac{e^{it}}{\sqrt{t}}\, dt\right] ,$$

which shows a change in the coefficient of the dominant term near the rim. Evaluation of the scattered amplitude as in (55)-(57), with the inclusion of $I^{(1)}(0)$ from (74), yields

$$(80) \qquad A^{(1)} \simeq -\frac{ika^2}{2} - \frac{a}{2} + \sqrt{\frac{a}{\pi k}}\, \exp\left[i(2ka-\frac{\pi}{4})\right] + O(\frac{1}{k}) ,$$

and thus

$$(81) \qquad \sigma = \pi a^2 \left(1 - \frac{2}{\sqrt{\pi}}\ \frac{\sin(2ka-\frac{\pi}{4})}{(ka)^{3/2}}\right) + O(ka)^{-2} .$$

We have thus succeeded in deriving an explicit correction to the geometrical cross-section, which is of lower order in 1/ka than that obtainable from the previous approximation, and should contain the principal diffraction effect when ka >> 1.

4. Variational calculation

It is natural to consider the applicability of variational techniques for the problem at hand, in view of their success for other aspects of scattering phenomena. Indeed, the transmission through a circular aperture in a soft screen has been calculated with high accuracy for the low-frequency range, $0 < ka < 4$, by use of a variational formulation which involves approximation to the aperture distribution of Ψ[10]. Another variational principle was developed by Levine and Schwinger[10], which is closer to the framework described in Section 3 in that it relates to the screen distribution of $\partial_z \Psi$; this principle also has the merit that it incorporates the refined Kirchhoff expression (18) as part of the scattering amplitude. The efficiency of variational procedures is contingent on the choice of trial functions, and here the dictates of similarity to the proper distribution and relative simplicity for computational purposes must be considered. That these are not always compatible is evidenced with half-plane screen distributions, which prove too complicated for exact evaluation of the integrals concerned. Thus, in earlier application, only simple forms of the screen distribution were employed, and these preclude an accurate estimate of the departure from geometrical behavior. It will be our purpose here to show that more precise, though complicated, distributions such as that of the half-plane may be exploited, at the expense of a consistent approximation to the integrals. We will, in fact, confirm by this approach the explicit correction to the geometrical cross-section given by (81).

A variational characterization of the scattered amplitude $\mathcal{A}$, based on the integral equation (25), can be extracted from the development in [10], where the directions of excitation and observation, as well as the shape of aperture, are left arbitrary. The result is

$$(82)\qquad A = A_K - \left(\frac{k}{2\pi}\right)^2 \left[\int_S K(\vec{\rho})\, \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\, K(\vec{\rho}\,')\, dS\, dS' - 2\int_S K(\vec{\rho})\, dS \int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\, dS' \right],$$

with a Kirchhoff term A_K of the form (18). The cross-section deduced from (82),

$$(83)\ \sigma = \sigma_K + \frac{k}{2\pi}\, \mathrm{Im}\left[\int_S K(\vec{\rho})\, \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\, K(\vec{\rho}\,') dS\, dS' - 2\int_S K(\vec{\rho}) dS \int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\, dS' \right],$$

likewise involves a Kirchhoff part, evaluated previously for the circular aperture. If the stationary nature of (83) is utilized in effecting a scale transformation of $K(\vec{\rho})$, we obtain the cross-section in a homogeneous form:

$$\sigma = \sigma_K - \frac{k}{2\pi}\,\mathrm{Im}\,\frac{\left(\int_S K(\vec{\rho})dS \int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|}dS'\right)^2}{\int_S K(\vec{\rho})\frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|}K(\rho')dSdS'} . \tag{84}$$

To improve the tractability of the integrals in (83) and (84), we modify the Green's function along lines suggested in the preceding section. Thus, for a circular aperture, recalling (28), (32), (33), (39), and (40), we put

$$\int_S K(\rho)dS \int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|}dS' = (2\pi)^2\, ia\int_0^\infty d\zeta\left\{\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right\}J_1(\zeta a)\int_a^\infty K(\rho)J_o(\zeta\rho)\rho d\rho$$

$$\simeq \frac{1}{\pi}(2\pi)^2 ia\int_0^\infty \frac{d\zeta}{\zeta}\left\{\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right\}\int_a^\infty K(\rho)\Big[\sin\zeta(\rho-a) + \cos\zeta(\rho+a)\Big]\sqrt{\rho/a}\; d\rho \tag{85}$$

$$= -2\pi a\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\epsilon}\,\frac{I(-\zeta)}{\sqrt{k^2-\zeta^2}} - 2\pi ia\int_0^\infty\frac{d\zeta}{\zeta}\left\{\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k}\right\}\left\{e^{2i\zeta a}I(-\zeta)+e^{-2i\zeta a}I(\zeta)\right\} .$$

Similarly,

$$\int_S K(\rho)\frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|}K(\rho')dSdS' = (2\pi)^2 i\int_0^\infty\frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}}\int_a^\infty K(\rho)J_o(\zeta\rho)\rho d\rho\int_0^\infty K(\rho')J_o(\zeta\rho')\rho' d\rho'$$

$$\simeq (2\pi)^2\,\frac{ia}{\pi}\int_0^\infty\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\int_0^\infty K(\rho)K(\rho')\Big[\cos\zeta(\rho-\rho') + \sin\zeta(\rho+\rho')\Big]\sqrt{\rho/a}\;\sqrt{\rho'/a}\; d\rho d\rho' \tag{86}$$

$$= 2\pi ia\int_{-\infty}^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}I(\zeta)I(-\zeta) + 2\pi a\int_0^\infty\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\left\{e^{2i\zeta a}\big(I(-\zeta)\big)^2 - e^{-2i\zeta a}\big(I(\zeta)\big)^2\right\} .$$

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A notable feature of this rearrangement is that the transforms of the screen distribution are put in evidence, as these take a simpler form in the half-plane problem than the coordinate functions. Using the transform $I^{(0)}(\zeta)$, which corresponds to incident wave excitation (with scale fixed by (51)), as an initial approximation, we have

$$(87) \qquad \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta+i\varepsilon}\, \frac{I^{(0)}(-\zeta)}{\sqrt{k^2-\zeta^2}} = -2i\int_{k}^{\infty} \frac{d\zeta}{\zeta}\, \frac{I^{(0)}(-\zeta)}{\sqrt{\zeta^2-k^2}} = \frac{\pi-2}{k^2}\,, \qquad \varepsilon \to 0$$

and

$$(88) \qquad \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}}\, I^{(0)}(\zeta) I^{(0)}(-\zeta) = 2\int_{k}^{\infty} \frac{d\zeta}{\zeta}\, \frac{I^{(0)}(-\zeta)}{\sqrt{\zeta^2-k^2}} = +i\,\frac{\pi-2}{k^2}\,.$$

Hence, the corresponding cross-section (83) becomes

$$(89) \qquad \sigma = \sigma_K + \frac{a}{k}\,\mathrm{Im}\left\{ 2I_1(2ka) - I_2(2ka) \right\},$$

where

$$(90) \qquad I_1(\alpha) = \int_0^{\infty} \frac{dx}{x^2} \left[\frac{1}{\sqrt{1-x^2}} - 1 \right] \left\{ e^{i\alpha x}(1-\sqrt{1+x}) - e^{-i\alpha x}(1-\sqrt{1-x}) \right\}$$

and

$$(91) \qquad I_2(\alpha) = \int_0^{\infty} \frac{dx}{x^2}\, \frac{1}{\sqrt{1-x^2}} \left\{ e^{i\alpha x}(1-\sqrt{1+x})^2 - e^{-i\alpha x}(1-\sqrt{1-x})^2 \right\}.$$

If the scale factor in $I^{(0)}$ were not known, we should appeal to the homogeneous form (84) of the cross-section, and deduce that

$$\sigma = \sigma_K + \frac{a}{k}(\pi-2)\ \mathrm{Im}\ \frac{\left(1 + \dfrac{I_1(2ka)}{\pi-2}\right)^2}{\left(1 + \dfrac{I_2(2ka)}{\pi-2}\right)} \ . \tag{92}$$

When $a >> 1$, the values of $I_1(\alpha)$ and $I_2(\alpha)$ are determined principally at $x \approx 1$ in the integral representations (90), (91). Writing $x = 1 - \frac{t}{2}$, we have for $\alpha >> 1$

$$I_1(\alpha) \simeq \frac{1}{2} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{t}} \left\{ \exp\left[i\alpha(1-\frac{t}{2})\right](1-\sqrt{2})^2 - \exp\left[-i\alpha(1-\frac{t}{2})\right] \right\} ,$$

and

$$I_2(\alpha) \simeq \frac{1}{2} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{t}} \left\{ \exp\left[i\alpha(1-\frac{t}{2})\right](1-\sqrt{2})^2 - \exp\left[-i\alpha(1-\frac{t}{2})\right] \right\} .$$

Since $\arg \sqrt{t}$ equals $\pi/2$ or 0 according as $t < 0$ or $t > 0$, we find for $\alpha > 0$

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{t}}\, e^{-i\alpha t/2} = 2\sqrt{\frac{2\pi}{\alpha}}\, e^{-i\pi/4} , \tag{93}$$

and

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{t}}\, e^{i\alpha t/2} = 0, \tag{94}$$

so that, for $\alpha >> 1$

$$I_1(\alpha) \simeq -(\sqrt{2}-1)\sqrt{\frac{2\pi}{\alpha}} \exp\left[i(\alpha - \frac{\pi}{4})\right] , \tag{95}$$

and

$$(96)\qquad I_2(\alpha) \simeq (\sqrt{2}-1)^2\sqrt{\frac{2\pi}{\alpha}}\exp\left[i(\alpha-\frac{\pi}{4})\right].$$

From the estimates (95) and (96), we find

$$(97)\qquad 2I_1(\alpha) - I_2(\alpha) \simeq -\sqrt{\frac{2\pi}{\alpha}}\exp\left[i(\alpha-\frac{\pi}{4})\right], \qquad \alpha \gg 1,$$

and thus (89) predicts

$$(98)\qquad \sigma \simeq \sigma_K - \frac{a}{k}\sqrt{\frac{\pi}{ka}}\sin(2ka-\frac{\pi}{4}), \qquad ka \gg 1.$$

Recalling the high-frequency development of σ_K (Eq. (21)), it is observed that the variational contribution to (98) equals that of the Kirchhoff term of the same order, and so

$$(99)\qquad \sigma \simeq \pi a^2\left(1-\frac{2}{\sqrt{\pi}}\;\frac{\sin(2ka-\frac{\pi}{4})}{(ka)^{3/2}}\right), \qquad ka \gg 1,$$

which agrees with the explicit terms in (81). The same asymptotic behavior is arrived at by employing the homogeneous form (92) of the cross-section, since its expansion in powers of I_1 and I_2, which functions approach zero with increasing argument, commences with the linear combination $2I_1 - I_2$ appearing in (89).

A detailed investigation of the behavior of I_1 and I_2 for large and small values of the argument is feasible, although we can circumvent this considerably in our further remarks about the related cross-sections. Indeed, since

$$
\begin{aligned}
I_2(\alpha) &= \int_0^\infty \frac{dx}{x^2}\left[\frac{1}{\sqrt{1-x^2}}-1\right]\left\{e^{i\alpha x}(1-\sqrt{1+x})^2 - e^{-i\alpha x}(1-\sqrt{1-x})^2\right\} \\
&\quad + \int_0^\infty \frac{dx}{x^2}\left\{e^{i\alpha x}(1-\sqrt{1+x})^2 - e^{-i\alpha x}(1-\sqrt{1-x})^2\right\} \qquad (100) \\
&= 2I_1(\alpha) + 2\int_0^\infty \frac{dx}{x}\left[\frac{1}{\sqrt{1-x^2}}-1\right]\cos\alpha x + \int_0^\infty \frac{dx}{x^2}\left\{e^{i\alpha x}(1-\sqrt{1+x})^2 - e^{-i\alpha x}(1-\sqrt{1-x})^2\right\},
\end{aligned}
$$

the combination $2I_1 - I_2$ appearing in the cross-section (89) is related to a new pair of integrals, one of which has previously been studied (cf. (61)). The leading term in the asymptotic development of this integral ($\alpha \gg 1$) suffices for reproducing (97), and it is easy to verify that the residual term in the cross-section (99), stemming from the other integral, is $O\left((ka)^{-2}\right)$. The relation (100) is also convenient for ascertaining the low-frequency ($ka \ll 1$) behavior of the cross-section; we note simply that whereas the transmission coefficient (cross-section relative to aperture area) should follow the Rayleigh law and decrease as $(ka)^4$, the forms (89) and (92) tend to infinity as $ka \to 0$.

We next examine the effect of including a secondary (grazing) wave term in the transform $I(\zeta)$. If $I(\zeta) = I^{(0)}(\zeta) + I^{(1)}(\zeta)$, then possible additional corrections to the geometrical cross-section of the order provided by $I^{(0)}$ alone (cf. (99)) are to be sought in the leading terms of (85) and (86) We find

$$
\delta\sigma = \frac{k}{2\pi}\,\mathrm{Im}\left[4\pi i a\int_{-\infty}^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\, I^{(0)}(\zeta)I^{(1)}(-\zeta) + 4\pi a\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\varepsilon}\,\frac{I^{(1)}(-\zeta)}{\sqrt{k^2-k^2}}\right],
$$

where the paths of integration are lines in the strip $|\mathrm{Im}\,\zeta| < \mathrm{Im}\,k$. Utilizing the explicit forms of $I^{(0)}$, $I^{(1)}$ (the latter to within a scale factor), we obtain

$$i\int_{-\infty}^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\,I^{(0)}(\zeta)I^{(1)}(-\zeta)+\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\varepsilon}\,\frac{I^{(1)}(-\zeta)}{\sqrt{k^2-\zeta^2}}$$

$$=\text{const.}\left[\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\varepsilon}\,\frac{1}{\sqrt{k^2-\zeta^2}}\,\frac{1}{\sqrt{1+\frac{\zeta}{k}}}-\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta}\,\frac{1}{\sqrt{k^2-\zeta^2}}\left(1-\sqrt{1-\frac{\zeta}{k}}\right)\frac{1}{\sqrt{1+\frac{\zeta}{k}}}\right]$$

$$=-2i(\text{const.})\left[\int_{k}^{\infty}\frac{d\zeta}{\zeta}\,\frac{1}{\sqrt{\zeta^2-k^2}}\,\frac{1}{\sqrt{1+\frac{\zeta}{k}}}-\int_{k}^{\infty}\frac{d\zeta}{\zeta}\,\frac{1}{\sqrt{\zeta^2-k^2}}\,\frac{1}{\sqrt{1+\frac{\zeta}{k}}}\right]=0,$$

where the crucial stage is reached after deformation of the integration contours in the upper half of the ζ-plane. Thus, $I^{(1)}$ does not contribute to the explicit correction term of (99) in the variational calculation, although as we have seen previously, it yields the entire contribution in a non-variational procedure. The variational technique affords us special advantage here, as the integral equation may be solved to the requisite order of approximation; however, the latter situation does not occur with the aperture formulation given in the following section, and then a variational procedure is appropriate.

Before concluding with screen formulations, we shall consider in more detail the Kirchhoff approach, briefly alluded to in Section 2, which is, in a sense, reciprocal to that discussed earlier (cf. Eq. (17)-(22)). Once again, the representations (3) and (4) are combined, but now (3) supplies an approximate wave function, in terms of the unperturbed aperture distribution, from which a normal derivative at the screen is determined; this normal derivative, together with (4), gives the transmitted field. We write

$$(101)\qquad \Psi_K(\vec{r})=2\int_A\frac{\partial}{\partial z'}\,G(\vec{r},\vec{r}\,')\Big|_{z'=0}\,dS',$$

and (cf. (23))

(102)
$$ikK(\vec{\rho}) = -\frac{\partial}{\partial z}\,\Psi_K(\vec{r})\Big|_{z=0} = -2\,\frac{\partial}{\partial z}\left[\int_A \frac{\partial}{\partial z'}\,G(\bar{r},\bar{r}')\Big|_{z=0}\,dS'\right]_{z\to 0 \text{ on } S}$$

$$= 2\,\frac{\partial^2}{\partial z^2}\left[\int_A G(\vec{r},\vec{\rho}')dS'\right]_{z=0} = -2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\int_A G(\vec{\rho},\vec{\rho}')dS',$$

using the differential equation

(103)
$$(\nabla^2 + k^2)\,G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}'),$$

and the relation

(104)
$$\partial_z G = -\partial_{z'} G.$$

With use of the integral representation (28), it follows that in the case of a circular aperture,

(105)
$$K(\rho) = -\frac{a}{k}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+k^2\right)F(\rho),$$

where

$$F(\rho) = \int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}}\,J_0(\zeta\rho)J_1(\zeta a).$$

For large ka we find (see Section 3) that the function $F(\rho)$ is given approximately by

(106)
$$F(\rho) \simeq \frac{1}{2k}\,\frac{1}{\sqrt{\rho a}}\int_{k(\rho-a)}^\infty H_0^{(1)}(t)dt,$$

and the dominant singularity of $K(\rho)$, $\rho \to a$, is therefore of the form $(\rho-a)^{-1}$, with imaginary coefficient. Since only the real part of $K(\rho)$ contributes to the cross-section,

$$\sigma = \pi a^2 - 2\pi \,\mathrm{Re} \int_a^\infty K(\rho)\rho d\rho, \tag{107}$$

an infinite value of σ is avoided. For the evaluation of (107), the real part of F is significant, namely

$$\mathrm{Re}\, F(\rho) = \int_0^k \frac{d\zeta}{\sqrt{k^2-\zeta^2}} J_0(\zeta\rho) J_1(\zeta a),$$

and it may be shown that

$$\sigma = \pi a^2 + \frac{\pi a}{k} \int_0^{2ka} \frac{J_1(x)}{x} dx \simeq \pi a^2 (1 + \frac{1}{ka}), \qquad k \gg 1. \tag{108}$$

The excessive magnitude of this correction to the geometrical cross-section, for $ka \gg 1$, also occurs in an approach to be described at the end of Section 5.

5. Aperture integral equation

From the expression (16) for the scattered amplitude $\mathcal{A}$ and the general theorem (13), we obtain

$$\sigma = \mathrm{Re} \int_A \Upsilon(\vec{\rho})\, dS \tag{109}$$

as the cross-section of an aperture in a soft screen at normal incidence. If we put

$$\Upsilon(\vec{\rho}) = 1 + \chi(\vec{\rho}), \tag{110}$$

the geometrical part of the cross-section is isolated, and we have

$$\sigma = A + \mathrm{Re} \int_A \chi(\vec{\rho}) dS, \tag{111}$$

the analogue of (27).

One might think that the aperture distribution $\Psi(\vec{\rho})$, defined in a bounded region, would be more easily characterized than a screen distribution $K(\vec{\rho})$, which covers an infinite region, and indeed for low frequencies this is the case. However, in the high-frequency case, the aperture field determination is more difficult, particularly when it is desired to account for the behavior both near the rim and near the aperture center. The field distribution near the aperture rim is readily obtained for the simple straight-edge geometry, in which case the screen and aperture problems are complementary. For the circular aperture, however, the use of a straight-edge approximation is necessarily inaccurate near the center; for better approximations, the complementarity is entirely lost. It is somewhat fortunate for this approach that the parts of the aperture distributions Ψ and χ near the rim are weighted more heavily in the cross-section calculation.

To bear out these contentions, and also to find the quantitative predictions of aperture formulations, we commence with the exact integro-differential equation in the aperture, which is a consequence of (3) and the property that in A

$$\partial_z \Psi = \partial_z \Psi^{inc} \quad .$$

This integro-differential equation is

$$(112) \qquad (\nabla_t^2 + k^2) \int_A \Psi(\vec{\rho}\,') \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho} - \vec{\rho}\,'|} dS' = 2\pi ik, \quad \vec{\rho} \text{ in } A \, .$$

Here ∇_t^2 is the transverse (to z) part of the Laplacian operator. When the aperture is circular, (112) can be written

$$(113) \qquad \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + k^2\right) \int_0^\infty \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}} J_0(\zeta\rho) \int_0^a \rho' J_0(\zeta P')\Psi(\rho')d\rho' = k, \qquad \rho < a$$

and integration yields an ordinary integral equation

$$\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} J_0(\zeta\rho) \int_0^a \rho' J_0(\zeta\rho')\Upsilon(\rho')d\rho' = \frac{1}{k} + \alpha J_0(k\rho) + \beta N_0(k\rho), \qquad \rho < a. \tag{114}$$

From the condition of finiteness at $\rho = 0$, we may require $\beta = 0$; the remaining integration constant α is chosen to ensure that $\Upsilon(a) = 0$. The function $\chi(\rho)$ (Eq. (110)) is described by the equation

$$\int_0^\infty \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}} J_0(\zeta\rho) \int_0^a \rho' J_0(\zeta\rho')\chi(\rho')d\rho' \tag{115}$$

$$= \alpha J_0(k\rho) + a\int_0^\infty d\zeta \left\{\frac{1}{k} - \frac{1}{\sqrt{k^2-\zeta^2}}\right\} J_0(\zeta\rho)J_1(\zeta a), \qquad \rho < a$$

since

$$\int_0^\infty J_0(\zeta\rho)\, J_1(\zeta a)\, d\zeta = \frac{1}{a}, \qquad \rho < a, \tag{116}$$

and $\chi(\rho)$ is subject to the boundary value

$$\chi(a) = -1. \tag{117}$$

Placing $\rho = 0$ in (115) relates α with an integral of $\chi(\rho)$; we have

$$\alpha = -a\int_0^\infty d\zeta \left\{\frac{1}{k} - \frac{1}{\sqrt{k^2-\zeta^2}}\right\} J_1(\zeta a) + \int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^a \rho J_0(\zeta\rho)\chi(\rho)d\rho \tag{118}$$

$$= -\frac{1}{k}\, e^{ika} - i\int_0^a e^{ik\rho}\chi(\rho)d\rho,$$

and this can be used as a consistency check on the value of α obtained via the boundary condition (117).

When $ka >> 1$, the important range is $\zeta \gtrsim k$ in the integrals of (115), and the Bessel function arguments are therefore large in comparison with the orders, except near the aperture center ($\rho = 0$). Disregarding the latter aspect, and approximating to the Bessel functions by asymptotic forms in the familiar manner, we find

$$\begin{aligned}\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^a \chi(\rho')\sqrt{\rho'}\left[\cos\zeta(\rho-\rho')+\sin\zeta(\rho+\rho')\right]d\rho' &= \pi\alpha\sqrt{\frac{2}{\pi k}}\cos(k\rho-\frac{\pi}{4}) \\ &+ \sqrt{a}\int_0^\infty \frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}}-\frac{1}{k}\right]\left[\sin\zeta(\rho-a)+\cos\zeta(\rho+a)\right], \qquad \rho < a.\end{aligned} \tag{119}$$

Introduce the coordinates x,x' such that $\rho = a - x$, $\rho' = a - x'$, and define

$$\phi(x) = \sqrt{1-\frac{x}{a}}\quad\chi(x), \tag{120}$$

whence

$$\begin{aligned}&\int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^a \phi(x')\left[\cos\zeta(x-x')+\sin\zeta(2a-x-x')\right]dx' \\ &= \alpha\sqrt{\frac{2\pi}{ka}}\cos\left(k(a-x)-\frac{\pi}{4}\right) - \int_0^\infty \frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}}-\frac{1}{k}\right]\left[\sin\zeta x-\cos\zeta(2a-x)\right], \qquad x > 0.\end{aligned} \tag{121}$$

This integral equation holds when x is positive and less than a. If we omit the integrands which contain the aperture radius a, and take the upper limit of the x' integral as infinite, the corresponding function $\bar{\phi}(x)$, $0 < x < \infty$, can be obtained exactly. $\bar{\phi}(x)$ will deviate from $\phi(x)$ near $x = a$, the aperture center, and actually (121) itself is not appropriate in this region, but we may still expect to find a reasonable cross-section, on the ground that the part of the aperture field near the rim ($x \approx 0$) is more important.

In the approximation contemplated, the function $\bar{\phi}(x)$ has a local character, and consequently on resolving the trigonometric term (with α as a coefficient) into plane waves, it is clear that the wave e^{-ikx}, describing interactions from distant parts of the aperture, must be without effect. We shall therefore omit this forcing term (this is justified in (Appendix 2)). Consider, then, the integral equation

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\int_{-\infty}^{\infty}\phi(x')e^{i\zeta(x-x')}dx' =$$

$$= \left\{\begin{matrix} \alpha\sqrt{\frac{\pi}{2ka}}\,e^{ikx}\exp\left[-i(ka-\frac{\pi}{4})\right], & x > 0 \\ \\ 0 & , x < 0 \end{matrix}\right\} + \frac{i}{2}\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\varepsilon}\,\frac{e^{i\zeta x}}{\sqrt{k^2-\zeta^2}} + \Lambda(x), \quad -\infty < x < \infty$$

with the stipulations

(123) $$\bar{\phi}(x) = 0, \quad x < 0; \qquad \Lambda(x) = 0, \quad x > 0.$$

The transform equation

(124) $$\pi\,\frac{\bar{\phi}(\zeta)}{\sqrt{k^2-\zeta^2}} = \alpha\sqrt{\frac{\pi}{2ka}}\exp\left[-i(ka-\frac{\pi}{4}\right]\frac{i}{k-\zeta} + \frac{i\pi}{\zeta+i\varepsilon}\,\frac{1}{\sqrt{k^2-\zeta^2}} + \Lambda(\zeta),$$

where

(125) $$\bar{\phi}(\zeta) = \int_{0}^{\infty} e^{-i\zeta x}\,\bar{\phi}(x)dx,$$

(126) $$\Lambda(\zeta) = \int_{-\infty}^{\infty} e^{-i\zeta x}\,\Lambda(x)\,dx,$$

holds in the strip $|\mathrm{Im}\,\zeta| < \mathrm{Im}\,k(=\varepsilon)$ of the ζ-plane. When Eq. (124) is rewritten as

(127) $$\frac{\bar{\phi}(\zeta)}{\sqrt{k-\zeta}} - \frac{i\alpha}{\sqrt{2\pi ka}}\,\frac{\sqrt{2k}\,e^{-i(ka-\frac{\pi}{4})}}{k-\zeta} - \frac{i}{\zeta+i\varepsilon}\left[\frac{1}{\sqrt{k-\zeta}} - \frac{1}{\sqrt{k+i\varepsilon}}\right] =$$

$$= \frac{i\alpha}{\sqrt{2\pi ka}}\exp\left[-i(ka-\frac{\pi}{4})\right]\frac{\sqrt{k+\zeta}-\sqrt{2k}}{k-\zeta} + \frac{i}{\zeta+i\varepsilon}\,\frac{1}{\sqrt{k+i\varepsilon}} + \frac{1}{\pi}\,\Lambda(\zeta)\sqrt{k+\zeta}\,,$$

the left- and right-hand members are regular in the half-planes $\mathrm{Im}\,\zeta < \mathrm{Im}\,k$, $\mathrm{Im}\,\zeta > -\mathrm{Im}\,k$ respectively, and we deduce from the vanishing of the entire function thus defined that

(128) $$\bar{\phi}(\zeta) = \frac{i}{\zeta}\left(1 - \sqrt{1-\frac{\zeta}{k}}\right) + \frac{i\alpha}{\sqrt{\pi a}}\,\frac{e^{-i(ka-\frac{\pi}{4})}}{\sqrt{k-\zeta}}\,,$$

with the inverse

(129) $$\bar{\phi}(x) = \frac{e^{i\pi/4}}{\sqrt{\pi}}\left\{\frac{e^{ikx}}{\sqrt{kx}} + i\int_{kx}^{\infty}\frac{e^{it}}{\sqrt{t}}\,dt\right\} - \frac{\alpha}{\pi\sqrt{a}}\,\frac{\exp\left[ik(x-a)+\frac{i\pi}{2}\right]}{\sqrt{x}}\,.$$

The terms involving $x^{-1/2}$ have to cancel and therefore*

(130) $$\alpha = -\sqrt{\pi\frac{a}{k}}\;e^{i(ka-\frac{\pi}{4})}\,,$$

whence

(131) $$\bar{\phi}(x) = \frac{i}{\sqrt{\pi}}\,e^{i\pi/4}\int_{kx}^{\infty}\frac{e^{it}}{\sqrt{t}}\,dt = -\frac{1}{\pi}\int_{1}^{\infty}\frac{dt}{t}\,\frac{e^{ikxt}}{\sqrt{t-1}}\,,$$

---

* Note that with this value of α, the corresponding inhomogeneous term in the integral equation (122) is independent of the aperture radius.

and

$$(132) \qquad \bar{\Phi}(\zeta) = \frac{i}{\zeta}\left(1 - \frac{1}{\sqrt{1-\frac{\zeta}{k}}}\right).$$

At $x = 0$ ($\rho = a$), both $\bar{\Phi}$ and $\bar{\chi}$ (obtained from (120)) take the value -1, in accordance with the boundary relation (117); for $x > 0$, $\bar{\Phi}$ is the local straight-edge function, decreasing as kx becomes larger, and due to the factor $\sqrt{1-\frac{x}{a}}$ which links $\bar{\Phi}$ and $\bar{\chi}$, the latter tends to infinity at the center of the aperture.

The expression (132) for $\bar{\Phi}(\zeta)$ can in fact be derived from the transform solution of the integral equation (45), since, as already pointed out, the function $F(x)$ appearing therein is related to $\Psi(x)$, namely

$$(133) \qquad F(x) = -\frac{2\pi}{k}\sqrt{1+\frac{x}{a}}\,\Psi(x), \qquad x < 0.$$

Now

$$F(\zeta) = -\frac{2\pi i}{k\zeta}\,\frac{1}{\sqrt{1+\frac{\zeta}{k}}},$$

so that the transform of $\sqrt{1+\frac{x}{a}}\;\Psi(x)$ is $\frac{i}{\zeta}\,\frac{1}{\sqrt{1+\frac{\zeta}{k}}}$, and subtracting the transform $\frac{i}{\zeta}$ of the incident field in $-\infty < x < 0$, we find

$$\bar{\Phi}(\zeta) = \frac{i}{\zeta}\left(\frac{1}{\sqrt{1+\frac{\zeta}{k}}} - 1\right);$$

this is substantially the result (132), except for reversal in sign of ζ, which stems from the fact that here the aperture domain corresponds to negative x.

The amplitude $\mathcal{A}$ can be written

$$(134) \qquad \mathcal{A} = -\frac{ika^2}{2} - ika\int_0^a \phi(x)\sqrt{1-\frac{x}{a}}\,dx,$$

and if the radical is approximated by $1 - \frac{x}{2a}$, and the integration limit thereafter is set equal to infinity, then using (132) we find (cf. (58))

$$\mathcal{A} \simeq -\frac{ika^2}{2} - \frac{a}{2} + \frac{3i}{16k}, \qquad ka \gg 1, \tag{135}$$

which implies a correction of order $(ka)^{-2}$ to the geometrical cross-section. The retention of a finite upper limit in this integration has the effect of replacing the $(ka)^{-2}$ correction with one of order $(ka)^{-3/2}$, where, however, the accompanying trigonometric factor depends on ka and not 2ka. Actually, the radical in (134) need not be approximated, and it can be verified that the cross-section resulting from the distribution $\bar{\phi}(x)$ is, asymptotically,

$$\sigma \simeq \pi a^2 \left(1 - \frac{\sin ka}{(ka)^2}\right), \qquad ka \gg 1.$$

It is interesting to ascertain the value of α which follows from the relation (118) when the function $\bar{\phi}(x)$ is involved; this value is

$$\alpha = -\frac{1}{k}\, e^{ika} - i\, e^{ika} \int_0^a \frac{dx}{\sqrt{1-\frac{x}{a}}}\, e^{-ikx}\, \bar{\phi}(x)$$

$$= -\frac{1}{k}\, e^{ika} + \frac{i}{\pi}\, e^{ika} \int_1^\infty \frac{du}{u} \frac{1}{\sqrt{u-1}} \int_0^a \frac{dx}{\sqrt{1-\frac{x}{a}}}\, e^{ik(u-1)x}.$$

For $x \neq 0$ and $kx \gg 1$, asymptotic evaluation of the u integral yields $\sqrt{\frac{\pi}{kx}}\, e^{i\pi/4}$, whence

$$\alpha \simeq -\frac{1}{k}\, e^{ika} + \frac{i}{\pi} \sqrt{\frac{\pi}{k}}\, \exp\left[i\left(ka + \frac{\pi}{4}\right)\right] \int_0^a \frac{dx}{\sqrt{x\left(1-\frac{x}{a}\right)}}$$

$$\simeq -\sqrt{\pi \frac{a}{k}}\, \exp\left[i\left(ka - \frac{\pi}{4}\right)\right], \qquad ka \gg 1,$$

which agrees with (130).

Further progress with the integral equations for χ or $\emptyset$ is difficult; an estimate has been made of the change brought about by extending the integration limit, whereby $\emptyset \to \overline{\emptyset}$, but this did not significantly improve the cross-section. We shall therefore turn next to a variational formulation of the scattered amplitude, and see how our present aperture distribution can be exploited. The variational expression reads

$$\mathcal{A} = -\frac{ik}{\pi}\int_A \Psi(\vec\rho)dS + \left(\frac{k}{2\pi}\right)^2\int_A \Psi(\vec\rho)\,\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}\,\Psi(\vec\rho\,')dSdS' \tag{136}$$
$$-\frac{1}{(2\pi)^2}\int_A \nabla\Psi(\vec\rho)\cdot\nabla'\Psi(\vec\rho\,')\,\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}\,dSdS',$$

with aperture functions Ψ that are required to vanish at the rim. Writing $\Psi = 1+\chi$, we have

$$\mathcal{A} = -\frac{ik}{\pi}\left(A+\int_A\chi(\vec\rho)dS\right)+\left(\frac{k}{2\pi}\right)^2\left[\int_A\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}dSdS' + 2\int_A\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}\chi(\vec\rho\,')dSdS'\right. \tag{137}$$
$$\left.+\int_A\chi(\vec\rho)\,\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}\,\chi(\vec\rho\,')dSdS'\right] - \frac{1}{(2\pi)^2}\int_A\nabla\chi(\vec\rho)\cdot\nabla'\ \ (\vec\rho\,')\ \frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}$$

whence

$$\sigma = 2A + 2\mathrm{Re}\int_A\chi(\vec\rho)dS - \frac{k}{2\pi}\,\mathrm{Im}\left[\int_A\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}dSdS' + 2\int_A\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}\chi(\vec\rho\,')dSdS'\right. \tag{138}$$
$$\left.+\int_A\chi(\vec\rho)\,\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}\,\chi(\vec\rho\,')dSdS'\right] + \frac{1}{2\pi k}\,\mathrm{Im}\int_A\nabla\chi(\vec\rho)\cdot\nabla'\chi(\vec\rho\,')\,\frac{e^{ik|\vec\rho-\vec\rho\,'|}}{|\vec\rho-\vec\rho\,'|}\,dSdS'.$$

We now particularize the various integrals in (138) containing the Green's function, for a circular domain A and a purely radial function $\chi(\rho)$. The first may be handled exactly (cf. (19), (20)):

$$(139)\qquad -\frac{k}{2\pi}\,\mathrm{Im}\int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\,dSdS' = -\pi a^2\left(1-\frac{J_1(2ka)}{ka}\right)$$

$$\simeq -\pi a\left(1-\frac{1}{\sqrt{\pi}}\,\frac{\sin(2ka-\frac{\pi}{4})}{(ka)^{3/2}}\right), \qquad ka \gg 1;$$

and in the other integrals the familiar Bessel function approximations are utilized. Thus

$$\int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\,\chi(\vec{\rho}\,')\,dSdS' = ia(2\pi)^2\int_0^\infty d\zeta\left\{\frac{1}{\sqrt{k^2-\zeta^2}}-\frac{1}{k}\right\}J_1(\zeta a)$$

$$\int_0^a \rho\,\chi(\rho)J_0(\zeta\rho)\,d\rho + i\frac{(2\pi)^2}{k}\int_0^a \rho\,\chi(\rho)\,d\rho$$

$$\simeq 2\pi a\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\varepsilon}\,\frac{\phi(-\zeta)}{\sqrt{k^2-\zeta^2}} - 2\pi i a\int_0^\infty\frac{d\zeta}{\zeta}\left[\frac{1}{\sqrt{k^2-\zeta^2}}-\frac{1}{k}\right]$$

$$\left[e^{2i\zeta a}\phi(\zeta)+e^{-2i\zeta a}\phi(-\zeta)\right] + i\,\frac{(2\pi)^2}{k}\int_0^a \rho\,\chi(\rho)\,d\rho,$$

where

$$(141)\qquad \phi(\zeta) = \int_0^a e^{-i\zeta x}\,\phi(x)\,dx.$$

Note that the integral of χ over the aperture is such as to cancel with the second term in (138). Further,

$$\int_A \chi(\rho) \frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|} \chi(\vec{\rho}')dSdS' = i(2\pi)^2 \int_0^\infty \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^a \rho\chi(\rho)J_o(\zeta\rho)d\rho \int_0^a \rho'\chi(\rho')J_o(\zeta\rho')d\rho'$$

$$(142) \qquad \simeq 2\pi ia \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \phi(\zeta)\phi(-\zeta) + 2\pi a \int_0^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \left[e^{2i\zeta a}\left(\phi(\zeta)^2 - e^{-2i\zeta a} \phi(-\zeta)\right)^2\right] .$$

The remaining integral, before approximation, is

$$\int_A \nabla\chi(\rho) \cdot \nabla'\chi(\rho') \frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|} dSdS'$$

$$(143) \qquad = i \sum_{n=0}^{\infty} (2-\delta)_{on}) \int_0^\infty \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}} \int_A dSdS' \frac{d\chi}{d\rho} \frac{d\chi}{d\rho'} \cos(\phi-\phi')J_n(\zeta\rho)J_n(\zeta\rho')\cos n(\phi-\phi')$$

$$= i(2\pi)^2 \int_0^\infty \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}} \int_0^a \frac{d\chi}{d\rho} \rho J_1(\zeta\rho)d\rho \int_0^a \frac{d\chi}{d\rho'} \rho' J_1(\zeta\rho')d\rho' ,$$

and to avoid a singularity in the radial integrals at the lower limit, the insertion of an asymptotic form of the Bessel function is deferred until after integration by parts, using the boundary value $\chi(a) = -1$, viz:

$$\int_0^a \frac{d\chi}{d\rho} \rho J_1(\zeta\rho)d\rho = -aJ_1(\zeta a) - \zeta \int_0^a \chi(\rho)\rho J_o(\zeta\rho)d\rho$$

$$\simeq -\sqrt{\frac{2a}{\pi\zeta}} \sin(\zeta a - \frac{\pi}{4}) - \sqrt{\frac{2\zeta}{\pi}} \int_0^a \chi(\rho) \sqrt{\rho} \cos(\zeta\rho - \frac{\pi}{4})d\rho$$

$$= -\sqrt{\frac{a\zeta}{2\pi}} \left\{ e^{i(\zeta a - \frac{\pi}{4})}\left(\phi(\zeta) - \frac{i}{\zeta}\right) + e^{-i(\zeta a - \frac{\pi}{4})}\left(\phi(-\zeta) + \frac{i}{\zeta}\right) \right\} .$$

Thus,

$$\left(\int_0^a \frac{d\chi}{d\rho}\,\rho J_1(\zeta\rho)\,d\rho\right)^2 \simeq \frac{a\zeta}{2\pi}\left[\exp\left[2i\left(\zeta a-\frac{\pi}{4}\right)\right]\left(\phi(\zeta)-\frac{i}{\zeta}\right)^2+2\left(\phi(\zeta)-\frac{i}{\zeta}\right)\left(\phi(-\zeta)+\frac{i}{\zeta}\right)\right.$$

$$\left.+\exp\left[-2i\left(\zeta a-\frac{\pi}{4}\right)\right]\left(\phi(-\zeta)+\frac{i}{\zeta}\right)^2\right],$$

and

$$\int_A \nabla\chi(\rho)\cdot\nabla'\chi(\rho')\,\frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\,dS\,dS'$$

$$\simeq 2\pi ia\int_{-\infty}^{\infty}\frac{\zeta^2 d\zeta}{\sqrt{k^2-\zeta^2}}\left(\phi(\zeta)-\frac{i}{\zeta}\right)\left(\phi(-\zeta)+\frac{i}{\zeta}\right)+2\pi a\int_0^{\infty}\frac{\zeta^2 d\xi}{\sqrt{k^2-\zeta^2}} \tag{144}$$

$$\left\{e^{2i\zeta a}\left(\phi(\zeta)-\frac{i}{\zeta}\right)^2-e^{-2i\zeta a}\left(\phi(-\zeta)+\frac{i}{\zeta}\right)^2\right\}.$$

Collecting results, we have

$$\sigma \simeq \pi a^2\left(1+\frac{1}{\sqrt{\pi}}\,\frac{\sin\left(2ka-\frac{\pi}{4}\right)}{(ka)^{3/2}}\right)$$

$$-\frac{k}{\pi}\,\mathrm{Im}\left\{2\pi a\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\varepsilon}\,\frac{\phi(-\zeta)}{\sqrt{k^2-\zeta^2}}+\pi ia\int_{-\infty}^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\left[\phi(\zeta)\phi(-\zeta)-\frac{\zeta^2}{k^2}\left(\phi(\zeta)-\frac{i}{\zeta}\right)\left(\phi(-\zeta)+\frac{i}{\zeta}\right)\right]\right\}$$

$$+ka\,\mathrm{Im}\left\{\int_0^{\infty}d\zeta\,e^{2i\zeta a}\left[\frac{2i\phi(\zeta)}{\zeta}\left(\frac{1}{\sqrt{k^2-\zeta^2}}-\frac{1}{k}\right)-\frac{(\phi(\zeta))^2}{\sqrt{k^2-\zeta^2}}+\frac{\zeta^2}{k^2}\,\frac{\left(\phi(\zeta)-\frac{i}{\zeta}\right)^2}{\sqrt{k^2-\zeta^2}}\right]\right. \tag{145}$$

$$\left.+\int_0^{\infty}d\zeta\,e^{-2i\zeta a}\left[\frac{2i\phi(-\zeta)}{\zeta}\left(\frac{1}{\sqrt{k^2-\zeta^2}}-\frac{1}{k}\right)+\frac{(\phi(-\zeta))^2}{\sqrt{k^2-\zeta^2}}-\frac{\zeta^2}{k^2}\,\frac{\left(\phi(-\zeta)+\frac{i}{\zeta}\right)^2}{\sqrt{k^2-\zeta^2}}\right]\right\}.$$

At this juncture, we have to introduce explicit forms of ϕ, and it is first of all appropriate to utilize the approximation $\bar{\phi}$, corresponding to infinite 'aperture' domain. Now

$$\int_0^\infty \frac{d\zeta}{\zeta+i\varepsilon}\,\frac{\bar{\phi}(-\zeta)}{\sqrt{k^2-\zeta^2}} = -2i\int_k^\infty \frac{d\zeta}{\zeta}\,\frac{\bar{\phi}(-\zeta)}{\sqrt{\zeta^2-k^2}} = -2\int_k^\infty \frac{d\zeta}{\zeta^2}\left(1-\frac{1}{\sqrt{1+\frac{\zeta}{k}}}\right)\frac{1}{\sqrt{\zeta^2-k^2}}$$

is real and therefore does not contribute to the cross-section. The same conclusion applies to the integral

$$i\int_{-\infty}^\infty \frac{d\zeta}{\sqrt{k^2-\zeta^2}}\left[\bar{\phi}(\zeta)\bar{\phi}(-\zeta)-\frac{\zeta^2}{k^2}\left(\phi(\zeta)-\frac{i}{\zeta}\right)\left(\phi(-\zeta)+\frac{i}{\zeta}\right)\right]$$

$$= i\int_{-\infty}^\infty \frac{d\zeta}{\zeta^2}\,\frac{1}{\sqrt{k^2-\zeta^2}}\left[1-\frac{1}{\sqrt{1-\frac{\zeta}{k}}}-\frac{1}{\sqrt{1+\frac{\zeta}{k}}}-\sqrt{1-\frac{\zeta^2}{k^2}}\right],$$

because its branch points and poles at $\zeta = \pm k$ are such that on deforming the path of integration in either half-plane, a real value results. In its simplified version, then, (145) appears as

$$\sigma \simeq \pi a^2\left(1+\frac{1}{\sqrt{\pi}}\,\frac{\sin(2ka-\frac{\pi}{4})}{(ka)^{3/2}}\right)+\frac{a}{k}\,\mathrm{Im}\left[\int_0^\infty \frac{dx}{x^2}\,\frac{e^{2ikax}}{\sqrt{1-x^2}}\left(x-2\sqrt{1+x}+2\sqrt{1-x^2}\right)\right.$$

(146)

$$\left.+\int_0^\infty \frac{dx}{x^2}\,\frac{e^{-2ikax}}{\sqrt{1-x^2}}\left(x+2\sqrt{1-x}-2\sqrt{1-x^2}\right)\right],$$

with an obvious change of variable from ζ to x. An asymptotic evaluation of the latter integrals, for ka >> 1, yields

(147) $$\sigma \simeq \pi a^2\left(1-2\left[\sqrt{2}-1\right]\frac{\sin(2ka-\frac{\pi}{4})}{(ka)^{3/2}}\right), \qquad ka \gg 1,$$

where the numerical coefficient, $2(\sqrt{2}-1) = 0.83$, of the second term is to be compared with the values 1 and 2 obtained by the refined Kirchhoff and screen approaches respectively. The 'finite' transform $\phi(\zeta)$, (141), obtained by integrating $\bar{\phi}(x)$, namely

$$(148) \qquad \int_0^a e^{-i\zeta x}\,\bar{\phi}(x)dx = \frac{i}{\zeta} - \frac{e^{i\pi/4}}{\zeta\sqrt{\pi}}\,\frac{1}{\sqrt{1-\frac{\zeta}{k}}}\int_0^{(k-\zeta)a}\frac{e^{it}}{\sqrt{t}}dt - \frac{e^{-i\zeta a+i\pi/4}}{\zeta\sqrt{\pi}}\int_{ka}^{\infty}\frac{e^{it}}{\sqrt{t}}dt,$$

can be employed in the cross-section calculation, but this does not reduce the numerical discrepancy. It would seem that since the aperture distribution is accurate near the rim only, the dimension of the aperture is effectively accounted for by the phase factors $e^{\pm 2i\zeta a}$ of (145).

When one seeks the result of including in the aperture distribution a part which arises from plane wave excitation (e^{-ikx}), it is the variational formulation which impedes progress; this stems from the fact that the class of admissible trial functions χ or ϕ is required to take a given value at the rim. Some modification of the formulation may prove advantageous, but will not be considered here.

A few concluding remarks about the integral equation (114) are appropriate. This equation may be handled trivially in the approximation whereby the radical $\sqrt{k^2-\zeta^2}$ is replaced by k; the Green's function is then a delta function, and the solution Ψ which vanishes at $\rho = a$ and remains finite at $\rho = 0$ is (cf. (31))

$$(149) \qquad \Psi(\rho) = 1 - \frac{J_0(k\rho)}{J_0(ka)};$$

from (109), we deduce the corresponding cross-section:

$$(150) \qquad \sigma = \pi a^2 - \frac{a}{k}\,\frac{J_1(ka)}{J_0(ka)} \simeq \pi a^2\left(1 - \frac{1}{\pi ka}\tan(ka-\frac{\pi}{4})\right), \qquad ka \gg 1.$$

Note that the wave function (149) tends to zero in a linear fashion near the rim, while at the aperture center the unperturbed part is generally masked by large oscillations of the remainder, when ka >> 1. Furthermore, if for the inverse radical we write

$$\left(\frac{1}{\sqrt{k^2-\zeta^2}}-\frac{1}{k}\right)+\frac{1}{k},$$

(114) can be converted to an integral equation of the second kind, capable of iteration. It appears, however, that successive iterations have some effect on the first correction to the geometrical cross-section, and hence the scheme is not really practical.

Acknowledgments

I am greatly indebted to Professors Keller, Marcuvitz and Schwinger for valuable discussions. A part of this work was completed during tenure of a fellowship of the John Simon Guggenheim Memorial Foundation.

Appendix 1

In Section 3, the screen integral equation (25), namely

$$\text{(A1)}\qquad \int_S K(\vec{\rho}\,') \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|} dS' = \int_A \frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|} dS', \qquad \vec{\rho} \text{ in } S$$

has been considered, with the help of an integral representation (28) for its kernel (Green's function). Here we wish to indicate another approach to the integral equation and cross-section calculation. Again, the circular domain A is assumed, and if ρ, ϕ are plane polar coordinates with origin at the center, then $K = K(\rho)$, while

$$\text{(A2)}\qquad |\vec{\rho}-\vec{\rho}\,'| = \sqrt{\rho^2+\rho'^2-2\rho\rho' \cos(\phi-\phi')} = \sqrt{(\rho-\rho')^2+4\rho\rho' \sin^2 \frac{\phi-\phi'}{2}} \ .$$

The straight-edge approximation (42) to the integral equation (A1) follows from the restriction $|\phi'-\phi| \ll 1$, which implies a local character of the screen distribution. If we write $\xi = \phi' - \phi$ and set

$$\sin^2 \frac{\phi' - \phi}{2} = \frac{1}{4} \xi^2,$$

the angle integrals in (A1) become

$$\int_{|\xi|<\Delta} \frac{e^{ik\sqrt{(\rho-\rho')^2+\rho\rho'\xi^2}}}{\sqrt{(\rho-\rho')^2+\rho\rho'\xi^2}} d\xi,$$

and for $k \to \infty$, this can be estimated by the corresponding infinite integral

$$\text{(A3)}\qquad \int_{-\infty}^{\infty} \frac{\exp\left[ik\sqrt{(\rho-\rho')^2 + \rho\rho'\xi^2}\ \right]}{\sqrt{(\rho-\rho')^2 + \rho\rho'\xi^2}} d\xi = \frac{\pi i}{\sqrt{\rho\rho'}} \ H_0^{(1)}(k|\rho-\rho'|) \ .$$

Hence, (A1) takes the form

$$\text{(A4)} \qquad \int_a^\infty \sqrt{\rho'}\, K(\rho') H_0^{(1)}(k|\rho-\rho'|)d\rho' = \int_0^a \sqrt{\rho'}\, H_0^{(1)}(k|\rho-\rho'|)d\rho', \qquad \rho > a$$

or, using the integral representation

$$\text{(A5)} \qquad H_0^{(1)}(k|\rho-\rho'|) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}}\, e^{i\zeta(\rho-\rho')},$$

we have

$$\text{(A6)} \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_a^\infty \sqrt{\rho'}\, K(\rho')\, e^{i\zeta(\rho-\rho')} d\rho' = \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}}\, e^{i\zeta\rho} \int_0^a \sqrt{\rho'}\, e^{-i\zeta\rho'} d\rho', \qquad \rho > a.$$

The inhomogeneous term in (A6) requires further approximation to secure agreement with Eq. (42), and we note first that it can be written equally well as

$$\int_{-\infty}^{\infty} d\zeta \left\{ \frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k} \right\} e^{i\zeta\rho} \int_0^a \sqrt{\rho'}\, e^{-i\zeta\rho'}\, d\rho', \qquad \rho > a,$$

since the domains of ρ and ρ' are distinct. Now, the significant values of ζ are of order k, and asymptotic development for the ρ' integral yields

$$\int_0^a \sqrt{\rho'}\, e^{-i\zeta\rho'}\, d\rho' = i \frac{d}{d\zeta} \int_0^a \frac{e^{-i\zeta\rho'}}{\sqrt{\rho'}}\, d\rho'$$

$$\simeq i \frac{d}{d\zeta} \left[\frac{\sqrt{\pi}\, e^{-i\pi/4}}{\sqrt{\zeta}} - \frac{1}{i\zeta} \frac{e^{-i\zeta a}}{\sqrt{a}} + \frac{1}{2i\zeta} \int_a^\infty \frac{e^{-i\zeta\rho'}}{(\rho')^{3/2}}\, d\rho' \right]$$

$$\simeq i \frac{\sqrt{a}}{\zeta}\, e^{-i\zeta a} - i \frac{\sqrt{\pi}\, e^{-i\pi/4}}{2\zeta^{3/2}} + O\left(\frac{1}{\zeta^2}\right).$$

Thus, retaining only the leading term, (A6) becomes

$$(A7)\quad \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{k^2-\zeta^2}} \int_a^{\infty} \sqrt{\rho'/a}\, K(\rho') e^{i\zeta(\rho-\rho')} d\rho' = -\int_{-\infty}^{\infty} \frac{d\zeta}{\zeta} \left[\frac{1}{\sqrt{k^2-\zeta^2}} - \frac{1}{k} \right] \sin\zeta(\rho-a), \qquad \rho > a,$$

when the odd part of the integrand on the right-hand side of (A6) is dropped. The change of variable $\rho = a + x$, and use of (39), (40) bring (A7) to the form (42).

Next, we take up the cross-section calculation, with the aim of utilizing the straight-edge distribution (A7), in the variational form

$$(A8)\quad \sigma = \sigma_K + \frac{k}{2\pi}\, \mathrm{Im} \left[\int_S K(\rho) \frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|} K(\rho') dS dS' - 2 \int_S \frac{e^{ik|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|} dS' \right] .$$

To evaluate the first of two angle integrals (ϕ, ϕ') of the Green's function, we now write

$$(A9)\quad \oint d\phi \, \frac{\exp\left[ik \sqrt{(\rho-\rho')^2 + 4\rho\rho' \sin^2 \frac{\phi-\phi'}{2}} \right]}{\sqrt{(\rho-\rho')^2 + 4\rho\rho' \sin^2 \frac{\phi-\phi'}{2}}}$$

$$= \left[\int_{|\phi-\phi'|<\Delta} d\phi + \int_{|\phi-\phi'|>\Delta} d\phi \right] \frac{\exp\left[ik \sqrt{(\rho-\rho')^2 + 4\rho\rho' \sin^2 \frac{\phi-\phi'}{2}} \right]}{\sqrt{(\rho-\rho')^2 + 4\rho\rho' \sin^2 \frac{\phi-\phi'}{2}}},$$

and adopt the previous estimates for the range $|\phi-\phi'| < \Delta$, namely, $\frac{\pi i}{\sqrt{\rho\rho'}} H_0^{(1)}(k|\rho-\rho'|)$. Since in the remaining interval $|\phi-\phi'| > \Delta$ the trigonometric function is bounded away from zero, and since k is large there, the radical in this interval is approximated by $2\sqrt{\rho\rho'}\, |\sin \frac{\phi-\phi'}{2}|$.

Also

$$\int_{|\xi|>\Delta} d\xi \, \frac{\exp\left[2ik\sqrt{\rho\rho'}\,\left|\sin\frac{\xi}{2}\right|\right]}{2\sqrt{\rho\rho'}\,\left|\sin\frac{\xi}{2}\right|} = 2\int_{0}^{\pi} d\xi \, \frac{\exp\left[2ik\sqrt{\rho\rho'}\,\sin\frac{\xi}{2}\right]}{2\sqrt{\rho\rho'}\,\sin\frac{\xi}{2}} - \int_{-\Delta}^{\Delta} d\xi \, \frac{\exp\left[ik\sqrt{\rho\rho'}\,\xi\right]}{\sqrt{\rho\rho'}\,\xi}$$

$$= 2\int_{0}^{\pi} d\xi \left[\frac{\exp\left[2ik\sqrt{\rho\rho'}\,\sin\frac{\xi}{2}\right]}{2\sqrt{\rho\rho'}\,\sin\frac{\xi}{2}} - \frac{\exp\left[ik\sqrt{\rho\rho'}\,\xi\right]}{\sqrt{\rho\rho'}\,\xi}\right] + O\left(\frac{1}{k}\right).$$

The function

$$P(\alpha) = \int_{0}^{\pi} d\xi \left[\frac{\exp\left(2i\alpha\sin\frac{\xi}{2}\right)}{2\sin\frac{\xi}{2}} - \frac{e^{i\alpha\xi}}{\xi}\right]$$

can be shown to have the asymptotic form

$$P(\alpha) \simeq \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,\exp\left[i\left(2\alpha - \frac{\pi}{4}\right)\right] + O\left(\frac{1}{\alpha}\right),$$

and consequently

$$\text{(A10)} \qquad \int_{|\xi|>\Delta} d\xi \, \frac{\exp\left[2ik\sqrt{\rho\rho'}\,\left|\sin\frac{\xi}{2}\right|\right]}{2\sqrt{\rho\rho'}\,\left|\sin\frac{\xi}{2}\right|} \simeq \sqrt{\frac{\pi}{k}}\,\frac{\exp\left[i\left(2k\sqrt{\rho\rho'} - \frac{\pi}{4}\right)\right]}{(\rho\rho')^{3/4}}.$$

Thus

$$\int_S K(\rho)\,\frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\,K(\rho')dSdS' \simeq \pi i(2\pi)\int_a^\infty K(\rho)H_0^{(1)}(k|\rho-\rho'|K(\rho')\,\sqrt{\rho\rho'}\;d\rho d\rho'$$

$$+\;2\pi\sqrt{\frac{\pi}{k}}\;\;e^{-i\pi/4}\int_a^\infty(\rho\rho')^{1/4}K(\rho)K(\rho')e^{2ik\sqrt{\rho\rho'}}d\rho d\rho'$$

(A11)

$$\simeq 2\pi ia\int_{-\infty}^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\,I(\zeta)I(-\zeta)\;+\;2\pi\sqrt{\frac{\pi a}{k}}\;\exp\left[i(2ka-\frac{\pi}{4})\right]\;\left(\int_0^\infty I(x)e^{ikx}dx\right)^2$$

$$=2\pi ia\int_{-\infty}^{\infty}\frac{d\zeta}{\sqrt{k^2-\zeta^2}}\,I(\zeta)I(-\zeta)+\;2\pi\sqrt{\frac{\pi a}{k}}\;\exp\left[i(2ka-\frac{\pi}{4})\right]\;(I(-k))^2,$$

where we have approximated $\exp\left[2ik\sqrt{\rho\rho'}\right]$ by $\exp\left[2ika(1+\frac{x+x'}{2a})\right]$, and also the geometrical factor $1+\frac{x}{a}$ by unity. Similarly,

$$\text{(A12)}\quad\int_S K(\rho)dS\int_A\frac{e^{ik|\vec{\rho}-\vec{\rho}\,'|}}{|\vec{\rho}-\vec{\rho}\,'|}\,dS' \simeq\; -2\pi a\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta+i\varepsilon}\,\frac{I(-\zeta)}{\sqrt{k^2-\zeta^2}}\;-2i\pi\,\frac{a}{k}\sqrt{\frac{\pi}{k}}\;\exp\left[i(2ka-\frac{\pi}{4})\right]I(-k).$$

If we employ the transform (of. (51))

$$I^{(0)}(\zeta)=\frac{i}{\zeta}\,(1-\sqrt{1-\frac{\zeta}{k}}\;),$$

then

$$\sigma \simeq \sigma_K + k\sqrt{\frac{\pi a}{k}}\ \mathrm{Im}\left[\exp\left[i(2ka-\frac{\pi}{4})\right]\left\{I^{(0)}(-k)^2 - \frac{2i}{k}\,I^{(0)}(-k)\right\}\right]$$

$$= \sigma_K - \frac{1}{k}\sqrt{\frac{\pi a}{k}}\ \sin(2ka - \frac{\pi}{4})$$

$$\simeq \pi a^2\left(1 - \frac{2}{\sqrt{\pi}}\ \frac{\sin(2ka - \frac{\pi}{4})}{(ka)^{3/2}}\right) \qquad ka \gg 1$$

which is the well-known result.

Appendix 2

We consider the integral equation

$$\text{(A13)} \qquad \int_0^\infty f(x')H_0^{(1)}(k|x-x'|)dx' = e^{-ikx}, \qquad x > 0$$

which may be identified as the equation determining the discontinuity in normal derivative of the wave function, $f(x)$, across a semi-infinite soft screen occupying the positive x-axis, excited by a grazing plane wave directed at the straight edge from the same side of the origin. Eq. (A13) is of the Wiener-Hopf type, and preliminary to solving it, we write an extended version:

$$\text{(A14)} \qquad \int_{-\infty}^{\infty} f(x')H_0^{(1)}(k|x-x'|)dx' = \left\{\begin{array}{ll} e^{-ikx-\delta x}, & x > 0 \\ \\ 0 \quad , & x < 0 \end{array}\right\} + g(x),$$

where it is assumed that

$$f(x) = 0, \qquad x < 0 \tag{A15}$$

and

$$g(x) = 0, \qquad x > 0; \tag{A16}$$

the convergence factor, $e^{-\delta x}$, $\delta > 0$, has been introduced in the plane wave term, and its relevance will be seen shortly. If both sides of this equation are multiplied by $e^{-i\zeta x}$, and integrated over all x, it follows with the help of the relation

$$H_o^{(1)}(k|x-x'|) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\zeta(x-x')}}{\sqrt{k^2-\zeta^2}}\, d\zeta,$$

that

$$2\,\frac{f(\zeta)}{\sqrt{k^2-\zeta^2}} = -\,\frac{i}{\zeta+k-i\delta} + g(\zeta). \tag{A17}$$

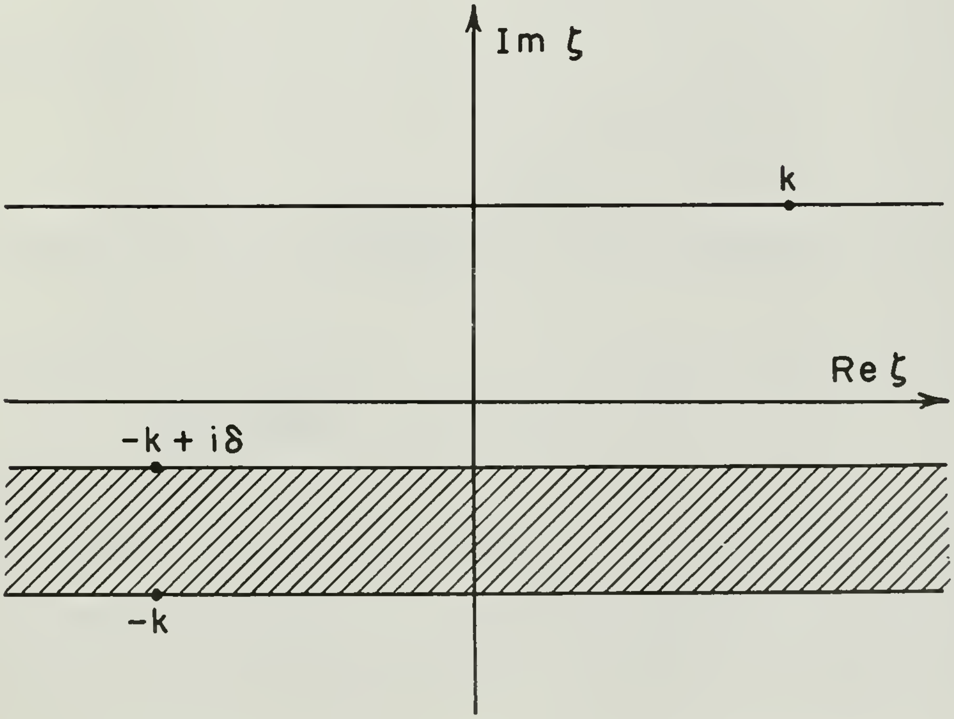

Figure 4

Now, if $\operatorname{Im} k = \varepsilon > 0$, the transform of the Hankel function is regular in the strip $|\operatorname{Im} \zeta| < \varepsilon$, the transforms of $f(x)$ and $e^{-ikx-\delta x}$ are regular in the (lower) half-plane $\operatorname{Im} \zeta < -\varepsilon+\delta$, while the transform of $g(x)$ is regular in the (upper) half-plane $\operatorname{Im} \zeta > -\varepsilon$. The common region of regularity defined thereby, namely $-\varepsilon+\delta > \operatorname{Im} \zeta > -\varepsilon$, owes its finite width to the parameter δ (see Fig. 4); with $\delta = 0$, the upper and lower half-planes would have only a common boundary with a singular point on it, which would make for complications. Rewriting the transform equation as

$$\text{(A18)} \qquad \frac{f(\zeta)}{\sqrt{k-\zeta}} + \frac{i}{2} \sqrt{i\delta} \; \frac{1}{\zeta+k-i\delta} = -\frac{i}{2} \, \frac{\sqrt{k+\zeta} - \sqrt{i\delta}}{\zeta+k-i\delta} + \frac{1}{2} \sqrt{k+\zeta} \; g(\zeta),$$

we can verify that the left- and right-hand sides are regular in the respective half-planes, so that an entire function is thereby defined throughout the ζ-plane. The entire function actually vanishes, and therefore

$$\text{(A19)} \qquad f(\zeta) = -\frac{i}{2} \sqrt{i\delta} \; \frac{\sqrt{k-\zeta}}{\zeta+k-i\delta} .$$

It is clear that both the transform $f(\zeta)$ and its inverse, $f(x)$, tend to zero with δ. However, in order to avoid an inconsistency with the original integral equation, we retain $\delta \neq 0$ and first calculate

$$\int_{-\infty}^{\infty} f(x')H_0^{(1)}(k|x-x'|)dx' = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\zeta x} f(\zeta)}{\sqrt{k^2-\zeta^2}} \, d\zeta$$

$$= -\frac{i}{2\pi} \sqrt{i\delta} \int_{-\infty}^{\infty} \frac{e^{i\zeta x}}{\sqrt{k+\zeta}} \, \frac{d\zeta}{\zeta+k-i\delta} ,$$

where the latter integration path lies within the strip of regularity. For $x > 0$, deformation in the upper half-plane captures the pole at $\zeta = -k+i\delta$, and the integral is simply $e^{-ikx-\delta x}$, which reverts to an undamped plane wave if $\delta \to 0$ and $\operatorname{Im} k = 0$.

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